1

Graph Theory (7BMAE1A) (Elective paper)

Definition: (Graph)

A Graph G consists of a pair $(V(G), E(G))$ where $V(G)$ is a non-empty finite set whose elements are called points or vertices and $E(G)$ is a set of unordered pairs of distinct elements of $V(G)$. The elements of $E(G)$ are called lines or edges of a graph G.

A graph with p vertices and q edges is called a (p, q) graph.

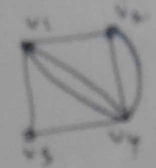
Defn: (Loop)

An edge of a graph that joins a vertex to itself is called loop or self loop.

Defn: (Multi graph)

If more than one edge joining two vertices are allowed, then the resulting graph is called a multi graph.

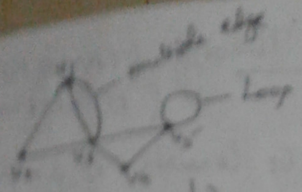
Ex: (G)



Defn: (Pseudo graph)

If loops and multiple edges are allowed in a graph, then the resulting graph is called pseudo graph.

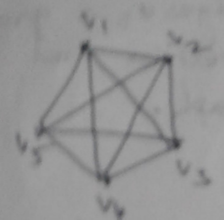
Ex:

(ii) 

Defn: (Complete graph)

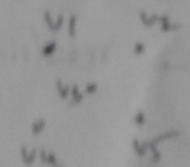
A graph in which any two distinct vertices are adjacent is called a complete graph. The complete graph with p vertices is denoted by K_p.

Ex:

K_5 : 

Defn: (Null graph)

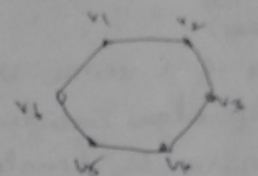
A graph whose edge set is empty is called a null graph or a totally disconnected graph.

Ex: 

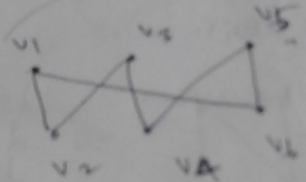
Defn: (Bigraph or Bipartite graph)

A graph G is called a bigraph or bipartite graph if V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge of G joins a vertex of V_1 to a vertex of V_2. (V_1, V_2) is called a bipartition of G.

Ex: C_6 :



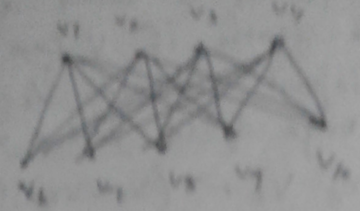
C_6 :



Defn: (Complete bipartite graph)

A graph G is called a complete bipartite graph if V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge of G joins a vertices of V_1 to a vertices of V_2. It is denoted by $K_{m,n}$. Here, V_1 contains m vertices and V_2 contains n vertices.

Ex: $K_{4,5}$

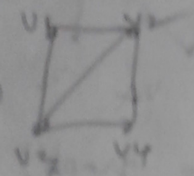


Defn: (Degree)

The degree of a vertex v_i in a graph G is the number of edges incident with v_i. The degree of v_i is denoted by $d_G(v_i)$ or $deg(v_i)$ or $d(v_i)$.

Ex:

(G)



$\Rightarrow$ $d(v_1) = 2$

$d(v_2) = 3$

$d(v_3) = 3$

$d(v_4) = 2$

maximum degree $\Delta(G) = \max\{\deg v / v \in V(G)\}$

$= 3$

minimum degree $\delta(G) = \min\{\deg v / v \in V(G)\}$

$= 2$

Defn: (Regular graph)

If all the vertices of G have the same degree r then $\delta(G) = \Delta(G) = r$. and in this case G is called a regular graph of degree r.

(5)

Theorem 1: (Hand shaking lemma)

The sum of the degrees of the vertices of a graph G is twice the number of edges i.e. $\sum_{i} \deg v_i = 2q$

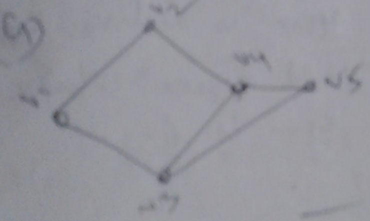
Proof: Each edge has two end points.

So, each edge will give degree two for both the end points of it.

If there are q number of edges in a graph G, then the sum of degrees of the vertices of a graph G is equal to 2q.

See the following example,

(G)



no. of edges in G = 6

Sum of all degrees of all degree vertices = 12.

—— x ——

Corollary: The number of vertices of odd degree in a graph is always even.

Proof: Let G be a graph with n vertices. Among those n vertices some of the vertices have odd degree and others have even degree

$$\therefore \sum_{i=1}^{n} d(v_i) = \sum_{odd} d(v_j) + \sum_{even} d(v_k)$$

$$\sum_{odd} d(v_j) = \sum_{i=1}^{n} d(v_i) - \sum_{even} d(v_k)$$

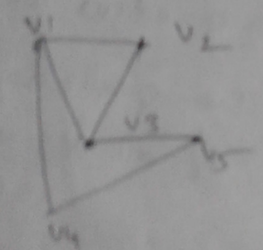
$$= 2(\text{no. of edges of } G) - \text{even number}$$

$$= \text{even}.$$

Hence, sum of vertices having odd degree is even.

See, the following example,

(G)



Here, v_1 and v_3 have odd degree whose sum is 6.

— x —

(7)

Problem: 1 Prove that $\delta \leq \frac{2q}{p} \leq \Delta$

Solution Let $V(G) = \{v_1, v_2, \cdots v_p\}$

We have $\delta \leq \deg v_i \leq \Delta$ for all i

Hence, $p\delta \leq \sum_{i=1}^{p} \deg v_i \leq p\Delta$

$\therefore p\delta \leq 2q \leq p\Delta$ (by the handshaking lemma)

$$\delta \leq \frac{2q}{p} \leq \Delta.$$

— x —

Problem: 2 Let G be a k-regular bigraph with bipartition (V_1, V_2) and $k > 0$. Prove that $|V_1| = |V_2|$.

Solution: Since, every line of G has one end in V_1 and other end in V_2, it follows that $\sum_{v \in V_1} d(v) = \sum_{v \in V_2} d(v) = q$.

Also, $d(v) = k$ for all $v \in V = V_1 \cup V_2$.

Hence, $\sum_{v \in V_1} d(v) = k|V_1|$ and

$\sum_{v \in V_2} d(v) = k|V_2|$

$\Rightarrow k|V_1| = k|V_2|$

Since, $k > 0$, we have $|V_1| = |V_2|$

Defn: (Sub graph)

A graph $H = (V_1, E_1)$ is called a sub-graph of $G = (V, E)$ if $V_1 \subseteq V$ and $E_1 \subseteq E$.

Defn: (Spanning Subgraph)

A subgraph H is called a spanning subgraph of G if $V_1 = V$.

Theorem: 1 The maximum number of edges among all p vertex graphs with no triangles is $[p^2/4]$.

Proof: The result can be easily verified for $p \leq 4$.

For $p > 4$, we will prove by Inductions separately for odd p and for even p.

part 1: For odd p

Suppose the result is true for all odd $p \leq 2n+1$.

Now, Let G be a (p, q) graph with $p = 2n+3$ and no triangles. If $q = 0$, then $q \leq [p^2/4]$. Hence, Let $q > 0$.

Let u and v be a pair of adjacent

vertices in G.

The subgraph $G' = G - \{u, v\}$ has $2n+1$ vertices and no triangles.

Hence by induction hypothesis,

$$q(G') \le \left[\frac{(2n+1)^2}{4}\right] = \left[\frac{4n^2+4n+1}{4}\right]$$

$$= \left[n^2+n+\frac{1}{4}\right] = n^2+n \quad \text{--- (1)}$$

Since, G has no triangles, no vertex of G' can be adjacent to both u and v in G. --- (2)

Now, edges in G are of three types.

(i) Lines of G' ($\le n^2+n$ in (1))

(ii) Lines between G' and $\{u, v\}$ ($\le 2n+1$ in (2))

(iii) Line uv.

Hence, $q \le (n^2+n)+(2n+1)+1 = n^2+3n+2$

$$= \frac{1}{4}(4n^2+12n+8)$$

$$= \frac{4n^2+12n+9}{4} - \frac{1}{4}$$

$$= \left[\frac{(2n+3)^2}{4}\right] = \left[\frac{p^2}{4}\right]$$

Also, for $p = 2n+3$, the graph $K_{n+1,\, n+2}$ has no triangles and has $(n+1)(n+2) = n^2+3n+2 = \left[\frac{p^2}{4}\right]$ lines.

Hence this maximum q is attained.

Part 2: For even p.

Suppose the result is true for all even $p \le 2n$.

Now, let G be a (p,q) graph with $p = 2n+2$ and no triangles.

As before, let u and v be a pair of adjacent points in G and

Let $G' = G - \{u, v\}$.

Now, G' has $2n$ points and no triangles.

Hence by hypothesis,

$$q(G') \le \left[\frac{(2n)^2}{4}\right] = n^2 \quad -(3)$$

Lines in G are of three types.

(i) Lines of G' $\le n^2$ in number by (3)

(ii) Lines between G' and $\{u, v\}$ $\le 2n$ in number by an argument similar (2)

(iii) Line uv.

(11)

Here, $q \leq n^2+2n+1 = (n+1)^2 = \frac{(2n+2)^2}{4} = \left[\frac{p^2}{4}\right]$

Hence, the result holds for even p also.

We see that for $p = 2n+2$,

$K_{n+1,n+1}$ is a $\left(p, \left[\frac{p^2}{4}\right]\right)$ graph without triangles.

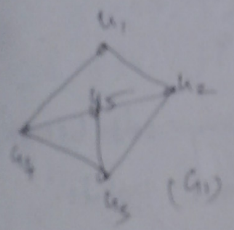
— x —

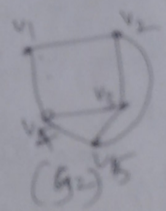
Defn: (Isomorphism)

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be isomorphic if there exists a bijection $f: V_1 \rightarrow V_2$ such that u, v are adjacent in G_1 iff $f(u), f(v)$ are adjacent in G_2.

If G_1 is isomorphic to G_2 we write $G_1 \cong G_2$.

Ex:





$f(u_i) = v_i$ is an isomorphism between these two graphs.

— x —

Defn: (Automorphism)

An isomorphism of a graph G onto itself is called an automorphism of G.

Defn: (Complement of G)

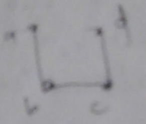
Let $G=(V,E)$ be a graph. The complement $\bar{G}$ of G is defined to be the graph which has V as its set of points and two points are adjacent in $\bar{G}$ iff they are not adjacent in G.

Defn: (Self complementary graph)

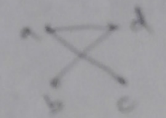
G is said to be a self complementary graph if G is isomorphic to $\bar{G}$.

Ex:

(G) :



$\bar{G}$:



Problem: 1 Prove that any self complementary graph has $4n$ or $4n+1$ points.

Solution: Let $G=(V(G), E(G))$ be a self complementary graph with p vertices.

Since, G is self complementary, G is isomorphic to $\bar{G}$,

So $|E(G)| = |E(\bar{G})|$

Also, $|E(G)| + |E(\bar{G})| = {}^{p}C_2 = \frac{p(p-1)}{2}$

$\therefore 2 \mid E(G) = \frac{p(p-1)}{2}$

$\therefore \lvert E(G) \rvert = \frac{p(p-1)}{4}$ is an integer.

Further, one of p or $p-1$ is odd.

Hence, p or $p-1$ is a multiple of 4.

p is of the form $4n$ or $4n+1$.

<u>Problem 2:</u> Prove that $\Gamma(G) = \Gamma(\bar{G})$

<u>Solution:</u> Let $f \in \Gamma(G)$ and let $u, v \in V(G)$

Then u, v are adjacent in $\bar{G} \Leftrightarrow u, v$ are not adjacent in G.

$\Leftrightarrow f(u), f(v)$ are not adjacent in G.

(Since, f is an automorphism of G)

$\Leftrightarrow f(u), f(v)$ are adjacent in $\bar{G}$.

Hence, f is an automorphism of $\bar{G}$.

$\therefore f \in \Gamma(\bar{G})$ and hence, $\Gamma(G) \subseteq \Gamma(\bar{G})$

Hly $\Gamma(\bar{G}) \subseteq \Gamma(G)$ so that $\Gamma(G) = \Gamma(\bar{G})$.

—x—

Ramsey Number

$r(m,n)$ is a Ramsey Number.

Problem 1: Prove that $r(m,n) = r(n,m)$

Solution: Let $r(m,n) = s$.

Let G be any graph on s points. Then $\bar{G}$ also has s points.

Since, $r(m,n) = s$, $\bar{G}$ has either K_m or $\bar{K}_n$ as an induced subgraph. Hence G has K_n or $\bar{K}_m$ as an induced subgraph.

Thus an arbitrary graph on s points contains K_n or $\bar{K}_m$ as an induced subgraph.

$\therefore r(n,m) \leq s$.

i.e., $r(n,m) \leq r(m,n)$.

Interchanging m and n we get $r(m,n) \leq r(n,m)$

Hence, $r(m,n) = r(n,m)$

— x —

Problem 2: Prove that $r(2,2) = 2$

Solution: Let G be a graph on 2 points.

Let $V(G) = \{u, v\}$.

Then u and v are either adjacent in G or adjacent in $\bar{G}$. Hence, G or $\bar{G}$ contains K_2. Thus if G is any graph on two points, then G or $\bar{G}$ contains K_2 and Clearly 2 is the least positive integer

(15) with this property.

Hence, $r(2,2)=2$

— x —

Independent sets and coverings

Defn: (Covering)

A covering of a graph $G=(V,E)$ is a subset K of V such that every edge of G is incident with a vertex in K.

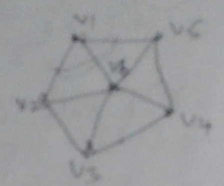
A covering K is called a minimum covering if G has no covering K' with $|K'|<|K|$.

The number of vertices in a minimum covering of G is called the covering number of G and is denoted by β.

Defn: (Independent set)

A subset S of V is called an independent set of G if no two vertices of S are adjacent in G. An independent set S is said to be maximum if G has no independent set S' with $|S'|>|S|$. The number of vertices in a maximum independent set is called the independence number of G and is denoted by α.

Example:



Consider the above graph.

$\{v_6\}$ is an independent set.

$\{v_1, v_5\}$ is a maximum independent set

$\{v_1, v_2, v_3, v_4, v_5\}$ is a covering and,

$\{v_2, v_6, v_4, v_5\}$ is a minimum covering.

— x —

Theorem:1 A set $S \subseteq V$ is an independent set of G if and only if $V - S$ is a covering of G.

proof: By definition, S is independent iff no two vertices of S are adjacent.

ie, iff every line of S is incident with atleast one point of $V-S$.

ie, iff $V-S$ is a covering of G.

— x —

Corollary: $\alpha + \beta = p$.

Proof: Let S be a maximum independent set of G and K be a minimum covering of G.

$\therefore |S| = \alpha$ and $|K| = \beta$.

Now, V-S is a covering of G and since K is a minimum covering of G

Hence, $|K| \leq |V-S|$ so that $\beta \leq p - \alpha$

$\therefore \beta + \alpha \leq p$ —(1)

Also, V-K is an independent set and S is a maximum independent set.

Hence, $|S| \geq |V-K|$ so that $\alpha \geq p - \beta$.

$\therefore \alpha + \beta \geq p$ —(2)

From (1) and (2), we get $\alpha + \beta = p$.

—x—

Defn: (Line covering)

A line covering of G is a subset L of X such that every vertex is incident with a line of L. The number of lines in a minimum line covering G is called the line covering number of G and is denoted by β'.

Defn: (Line independent set)
A set of line is called independent if no two of them are adjacent.
The number of lines in maximum independent set of lines is called the line independent number and is denoted by α'.

Result: $\alpha' + \beta' = p$.

proof: Let S be a maximum independent set of lines of G so that $|S| = \alpha'$.
Let M be a set of lines one incident line for each of the $p - 2\alpha'$ points of G not covered by any line of S.
Clearly $S \cup M$ is the line covering of G.

$$|S \cup M| \geq \beta'$$

$$\therefore \alpha' + p - 2\alpha' \geq \beta'$$

$$\therefore p \geq \alpha' + \beta' \quad \text{--- (1)}$$

Now, let T be a minimum line cover of G, so that $|T| = \beta'$.
T cannot have a line x both of whose ends are also incident with lines

(19) ~~for each of the p − 2α' points of G.~~ of T other than x. Hence G[T] the spanning subgraph of G induced by T, is the union of stars.

Hence each line of T is incident with at least one end point of G[T].

Let W be a set of endpoints of G[T] consisting of exactly one endpoint for each line of T.

Hence $|W| = |T| = \beta'$ and each star has exactly one point not in W.

Hence

$$p = |W| + (\text{number of stars in } G[T])$$

$$p = \beta' + (\text{number of stars in } G[T]) \quad \text{---(2)}$$

By choosing one line from each star of G[T], we get a set of independent lines of G, Hence $\alpha' \geq$ (number of stars in G[T])

Hence, (2) gives $p \leq \beta' + \alpha'$.

$\therefore$ by (1), $\alpha' + \beta' = p$.

Hence the proof.

—x—

Matrices:

Defn: (adjacency matrix)

Let $G = (V, E)$ be a (p, q) graph. Let $V = \{v_1, v_2, \ldots, v_p\}$. The $p \times p$ matrix $A = (a_{ij})$ where

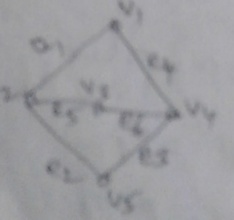
$$a_{ij} = \begin{cases} 1 & \text{if } v_i, v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

is called the adjacency matrix of the graph G.

For example, the adjacency matrix of the graph in the following figure is given by

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

(a)



(21) **Defn:** (Incidence matrix)

Let $G=(V,E)$ be a (p,q) graph.

Let $V=\{v_1, v_2, \ldots, v_p\}$ and $E=\{e_1, e_2, \ldots, e_q\}$

The $p\times p$ matrix $B=(b_{ij})$ where

$$b_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is incident with } e_j \\ 0 & \text{otherwise} \end{cases}$$

is called the incidence matrix of the graph.

For example, the incidence matrix of the graph given in figure is given below.

$$B = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{pmatrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

— x —

Defn: (Intersection graph)

The set of points V of $\Omega(F)$ is F itself and two points S_1, S_2 are adjacent if $i \neq j$ and $S_i \cap S_j \neq \phi$. A graph G is called an intersection graph on S if there exists a family F of subsets of S such that G is isomorphic to $\Omega(F)$.

Theorem: Every graph is an intersection graph.

Proof: Let $G = (V, E)$ be a graph. Let $V = \{v_1, \ldots, v_p\}$. Let $S = V \cup E$.

For each $v_i \in V$, let $S_i = \{v_i\} \cup \{x \in E / v_i \in x\}$.

Clearly $F = \{S_1, S_2, \ldots, S_p\}$ is a family of distinct nonempty subsets of S.

Further if v_i, v_j are adjacent in V then $v_i v_j \in S_i \cap S_j$ and hence $S_i \cap S_j \neq \phi$.

Conversely, if $S_i \cap S_j \neq \phi$ then the element common to $S_i \cap S_j$ is the line joining v_i and v_j so that v_i, v_j are adjacent in G. Thus $f : V \to F$ defined by $f(v_i) = S_i$ is an isomorphism of G to $\Omega(F)$.

Hence G is an intersection graph.

— x —

Defn: (Line graph)

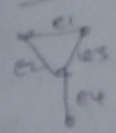
Let $G = (V, E)$ be a graph with $E \neq \phi$. Then E can be thought of as a family of 2 elements subsets of V.

The intersection graph $\Omega(E)$ is called the line graph of G and is denoted

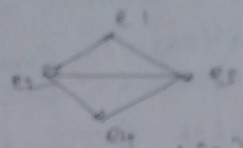
(3)

by $L(G)$. Thus the two points of $L(G)$ are the lines of G and two points in $L(G)$ are adjacent iff the corresponding lines are adjacent in G.

Example:



(G)



$L(G)$.

—×—

Theorem Let G be a (p,q) graph. Then $L(G)$ is a (q, q_L) graph where

$$q_L = \frac{1}{2}\left(\sum_{i=1}^{p} d_i^2\right) - q.$$

proof: By defn, number of points in $L(G)$ is p.

To find the number of edges in $L(G)$. Any two of the d_i lines incident with v_i are adjacent in $L(G)$ and hence we get $\frac{d_i(d_i-1)}{2}$ lines in $L(G)$.

Hence $q_L = \sum_{i=1}^{p} \frac{d_i(d_i-1)}{2}$

$= \frac{1}{2}\left(\sum_{i=1}^{p} d_i^2\right) - \frac{1}{2}\left(\sum_{i=1}^{p} d_i\right)$

$= \frac{1}{2}\left(\sum_{i=1}^{p} d_i^2\right) - \frac{1}{2}(2q)$ (by 1st lemma)

$= \frac{1}{2}\left(\sum_{i=1}^{p} d_i^2\right) - q$.

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Operations on Graphs

Definition: Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two graphs with $V_1 \cap V_2 = \phi$. We define

(i) The union $G_1 \cup G_2$ to be (V, E) where,

$$V = V_1 \cup V_2 \text{ and } E = E_1 \cup E_2$$

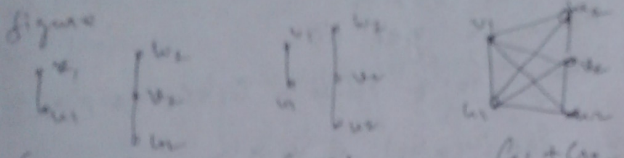
(ii) The sum $G_1 + G_2$ as $G_1 \cup G_2$ together with all the edges joining points of V_1 to points of V_2.

(iii) The product $G_1 \times G_2$ as having $V = V_1 \times V_2$ and $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent if $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2 or u_1 is adjacent to v_1 in G_1 and $u_2 = v_2$.

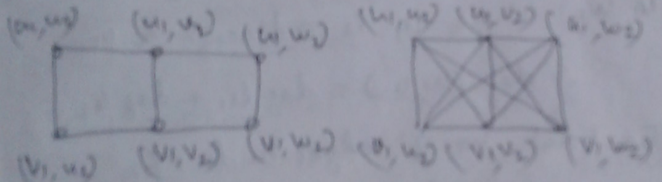
— x —

(iv) The composition $G_1[G_2]$ as having $V = V_1 \times V_2$ and $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent if u_1 is adjacent to v_1 in G_1 or ($u_1 = v_1$ and u_2 is adjacent to v_2 in G_2)

We illustrate these concepts in the following figure



(G_1) (G_2) $G_1 \cup G_2$ $G_1 + G_2$



$G_1 \times G_2$ $G_1[G_2]$.

Note: $\bar{K}_m + \bar{K}_n = K_{m,n}$

Theorem: Let G_1 be a (p_1, q_1) graph and G_2 be a (p_2, q_2) graph.

(i) $G_1 \cup G_2$ is a (p_1+p_2, q_1+q_2) graph
(ii) $G_1 + G_2$ is a $(p_1+p_2, q_1+q_2+p_1p_2)$ graph
(iii) $G_1 + G_2$ is a $(p_1p_2, q_1p_2+q_2p_1)$ graph
(iv) $G_1 \times G_2$ is a $(p_1p_2, p_1q_2+p_2^2 q_1)$ graph.
(v) $G_1[G_2]$

proof: (i) is obvious

(ii) Number of lines in $G_1 + G_2$

= no. of lines in G_1 + no. of lines in G_2

+ no. of lines joining points of V_1 to points of V_2.

$= q_1 + q_2 + p_1 p_2$.

Hence we get (ii).

(iii) clearly, no. of points in $G_1 \times G_2$ is $p_1 p_2$.

Now, let $(u_1, u_2) \in V_1 \times V_2$

The points adjacent to (u_1, u_2) are (u_1, v_2) where u_2 is adjacent to v_2 and (v_1, u_2), where v_1 is adjacent to u_1.

$\therefore \deg(u_1, u_2) = \deg u_1 + \deg u_2$.

The total no. of edges in $G_1 \times G_2$

$$= \frac{1}{2} \sum_{i,j} \deg(u_i) + \deg(v_j).$$

$$= \frac{1}{2} \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} (\deg u_i + \deg v_j)$$

where $u_i \in V_1, v_j \in V_2$

$$= \frac{1}{2} \sum_{i=1}^{p_1} \left(p_2 \deg u_i + \sum_{j=1}^{p_2} \deg v_j\right)$$

$$= \frac{1}{2} \sum_{i=1}^{p_1} (p_2 \deg u_i + 2q_2)$$

$$= \frac{1}{2} (p_2 2q_1 + p_1 2q_2) = p_2 q_1 + p_1 q_2$$

(iv) is left. —x—

Defn: (partition (or) degree sequence)

Let G be a (p, q) graph. The partition of 2q as the sum of the degrees of its points is called the partition or the degree sequence of the graph G.

Defn: (Graphical partition (or) graphical sequence)

A partition $P = (d_1, d_2, \ldots, d_p)$ of n into p parts is said to be a graphical partition or a graphical sequence if there exists a graph G whose points have degree d_i and G is called a realization of P.

Problem:1 Show that the partition $P = (7, 6, 5, 4, 3, 2)$ is not graphic.

Solution: Suppose P is graphic. Let G be a realization of P. Then G has six points. Hence the maximum degree of any point in G is 5 which is a contradiction.

Hence P is not graphic.

Problem:2 Show that the partition $P = (6, 6, 5, 4, 3, 3, 1)$ is not graphic.

Solution: Suppose P is graphic.

Let G be a realization of P.

(5)

Then G has seven points.
Since two points of G have degree 6, each of these two points is adjacent with every other point of G.
Hence the degree of each vertex in G is at least 2 so that G has no point of degree 1.

$\therefore$ p is not graphic.

— x —

Exercises:

(i) Find two non-isomorphic graphs with partition (2, 2, 2, 1, 1)

(ii) Show that the partition (7, 6, 5, 4, 3, 3, 1) is not graphical.

Theorem: A partition $P = (d_1, d_2, \dots, d_p)$ of an even number into p parts with $p-1 \geq d_1 \geq d_2 \geq \dots \geq d_p$ is graphical iff the modified partition $p' = (d_2 - 1, d_3 - 1, \dots, d_{d_1+1} - 1, d_{d_1+2}, \dots, d_p)$ is graphical.

Proof: Suppose p' is a graphical sequence.
Let G' be a graph with vertex set $\{v_2, v_3, \dots, v_p\}$ such that

$d(v_2) = d_2 - 1, \dots, d(v_p) = d_p$.

Let G be the graph obtained from G' by adding a new vertex v_1 and making it adjacent to $v_2, v_3, \ldots, v_{d_1+1}$.

Clearly the partition of G is P and hence P is a graphic sequence.

Conversely, suppose P is graphical.

Let $G = (V, E)$ be a realisation of P.

Let $V = \{v_1, v_2, \ldots, v_p\}$ with $\deg v_i = d_i$.

If v_1 is adjacent to $v_2, v_3, \ldots, v_{d_1+1}$, then $G - v_1$ is a realization of P'.

If the graph G does not have this property we will show that from G we can construct another realization of P having this property.

Hence assume that in G, v_1 is not adjacent to all the vertices $v_2, v_3, \ldots, v_{d_1+1}$.

Then there exist two vertices v_i and v_j such that $d_i > d_j$ and v_1 is adjacent with v_j but not adjacent with v_i.

Since, $d_i > d_j$, there exists a vertex v_k such that v_k is adjacent with v_i but not adjacent with v_j.

Let G' be the graph obtained from G by deleting the edges v_1v_j, v_jv_k and by adding the edges v_1v_k and v_jv_k. ~~Clearly G' be the graph obtained from G by deleting the edges v_1v_j, v_kv_x and~~

Clearly G' is also a realization of P in which v_1 is adjacent with v_i, but not with v_j. By repeating process we obtain a realization of P in which v_1 is adjacent to $v_2, v_3, \ldots, v_{d_1+1}$ and hence the theorem is proved.

Theorem: 2 If a partition $P = (d_1, d_2, \ldots, d_p)$ with $d_1 \geq d_2 \geq \cdots \geq d_p$ is graphical then $\sum_{i=1}^{p} d_i$ is even and $\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{p} \min\{k, d_i\}$ for $1 \leq k \leq p$.

proof: Let $G = (V, E)$ be a realization of P with $V = \{v_1, v_2, \ldots, v_p\}$ and $\deg v_i = d_i$. By theorem 1,

$$\sum_{i=1}^{p} d_i = 2q, \text{ which is even.}$$

(34) Now the sum $\sum_{i=1}^{k} d_i$ is the sum of the degrees of the vertices $v_1, v_2, \ldots, v_k$. It can be divided into two parts. The 1^{st} part being contribution to this sum by edges joining the points $v_1, v_2, \ldots, v_k$ and the 2^{nd} part being contribution to this sum by edges joining one of the points $v_{k+1}, v_{k+2}, \ldots, v_p$ with points of the set $\{v_1, v_2, \ldots, v_k\}$. The 1^{st} part is $\leq k(k-1)$.

The 2^{nd} part is $\leq \sum_{i=k+1}^{p} \min\{k, d_i\}$

Hence $\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{p} \min\{k, d_i\}$

Connectedness

Defn: (walk)
A walk of a graph G is an alternating sequence of points and edges $v_0, x_1, v_1, x_2, \ldots, v_{n-1}, x_n, v_n$ beginning and ending with points such that each edge x_i is incident with v_{i-1} and v_i.
The walk joins v_0 and v_n and it is called a $v_0 - v_n$ walk.

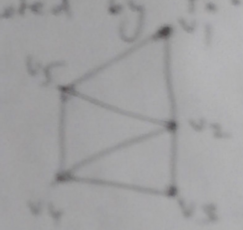
v_0 is called the initial point and v_n is called the terminal point of the walk.

Defn: (Trail)
A walk is called a trail if all its edges are distinct.

Defn: (path)
A walk is called a path if all its vertices are distinct. A path with n points is denoted by P_n.

Example:

(G)



In this above graph.

$v_1, v_2, v_3, v_4, v_2, v_1, v_2, v_5$ is a walk

$v_1, v_2, v_4, v_3, v_2, v_5$ is a trail but not a path

v_1, v_2, v_4, v_5 is a path.

Note: Every path is a trail and a trail need not be a path.

Defn: (cycle)

A $v_0 - v_n$ walk is called closed if $v_0 = v_n$.

A closed walk $v_0, v_1, v_2 \dots v_n = v_0$ in which $n \geq 3$ and $v_0, v_1, \dots v_{n-1}$ are distinct is called a cycle of length n.

The graph consisting of a cycle of length n is denoted by C_n.

— x —

Theorem:) In a graph G, any $u-v$ walk contains a $u-v$ path.

Proof: We prove the result by induction on the length of the walk.

Any walk of length 0 or 1 is obviously a path.

Now assume the result for all walks of length less than n.

Let $u = u_0, u_1, \dots u_n = v$ be a $u-v$ walk of length ~~less than~~ n.

If all the points of the walk are

distinct it is already a path.
If not there exists i and j such that $0 \le i < j \le n$ and $u_i = u_j$.
Now $u = u_0 \ldots u_i u_{j+1} \ldots u_n = v$ is a u-v walk of length less than n.
which is by induction hypothesis contains a $(u-v)$ path.

— x —

__Theorem:__ If $\delta \ge k$, then G has a path of length k.

__proof:__ Let v_1 be an arbitrary point.
Choose v_2 adjacent to v_1.
Since $\delta \ge k$, there exists atleast $k-1$ vertices other than v_1 which are adjacent to v_2.
Choose $v_3 \ne v_1$ such that v_3 is adjacent to v_2.
In general having chosen $v_1, v_2, \ldots v_i$ where $1 \le i \le k$ there exists a point $v_{i+1} \ne v_1, v_2, \ldots v_i$ such that v_{i+1} is adjacent to v_i. This process yields a path of length k in G.

Theorem 3: A closed walk of odd length contains a cycle.

Proof: Let $v = v_0, v_1, \ldots, v_n = v$ be a closed walk of odd length.

Hence $n \geq 3$. If $n = 3$, this walk is itself the cycle C_3 and hence the result is trivial.

Now assume the result for all walks of length less than n.

If the given walk of length n is itself a cycle there is nothing to prove.

If not, there exists two positive integers i & j such that $i < j$, $\{i, j\} \neq \{0, n\}$ and $v_i = v_j$.

Now $v_i, v_{i+1}, \ldots, v_j$ and $v = v_0, v_1, \ldots v_i, v_{j+1}, \ldots v_n = v$ are closed walks contained in the given walk and the sum of their lengths is n.

Since n is odd at least one of these walks is of odd length which by induction hypothesis contains a cycle.

—x—

Problem: If A is the adjacency matrix of a graph with $V = \{v_1, v_2, \dots, v_p\}$. Prove that for any $n \geq 1$ the (i,j)th entry of A^n is the number of $v_i - v_j$ walks of length n in G.

Solution: We prove the result by induction on n.

The number of $v_i - v_j$ walks of length 1

$$= \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

$= a_{ij}$

Hence the result is true for $n = 1$.

We now assume that the result is true for $n-1$.

Let $A^{n-1} = (a_{ij}^{(n-1)})$ so that $a_{ij}^{(n-1)}$ is number of $v_i - v_j$ walks of length $(n-1)$ in G.

Now, $A^{n-1} A = (a_{ij}^{(n-1)})(a_{ij})$

Hence, $(i,j)^{th}$ entry of $A^n = \sum_{k=1}^{p} a_{ik}^{(n-1)} a_{kj}$ —— (1)

Also, every $v_i - v_j$ walk of length n in G consists of a $v_i - v_k$ walk of

(37) length $n-1$ followed by a vertex v_j which is adjacent to v_k. Hence if v_j is adjacent to v_k then $a_{kj} = 1$ and $a_{ij}^{(n-1)} a_{kj}$ represents the number of $v_i - v_j$ walks of length n whose last edge is $v_k v_j$. Hence the right side of (1) gives the number of $v_i - v_j$ walks of length n in G.

Hence the proof.

Connectedness and Components

Defn: (Connected)

Two points u and v of a graph G are said to connected if there exists a $u-v$ path in G.

Defn: (Not Connected)

A graph which is not connected is said to be disconnected.

Ex: The union of two graphs is disconnected.

Components

Let G_i denote the induced subgraph of G with vertex set V_i. Clearly the subgraph $G_1, G_2, \ldots, G_n$ are connected and are called the components of G.

Theorem: A graph G with p points and $\delta \geq \frac{p-1}{2}$ is connected.

Proof: Suppose G is not connected. Then G has more than one component. Consider any component $G_1 = (V_1, E_1)$ of G. Let $v_1 \in V_1$. Since $\delta \geq \frac{p-1}{2}$ there exists at least $\frac{p-1}{2}$ points in G_1 adjacent to v_1 and hence V_1 contains at least

$$\frac{p-1}{2} + 1 = \frac{p+1}{2} \text{ points.}$$

Thus each component of G contains at least $\frac{p+1}{2}$ points and G has atleast two components. Hence the number of points in $G \geq p+1$

Which is contradiction

Hence G is connected.

—x—

Theorem: 5 A graph G is connected iff for any partition of V into subsets V_1 and V_2 there is an edge of G joining a vertex of V_1 to a vertex of V_2.

Proof: Suppose G is connected

Let $V = V_1 \cup V_2$ be a partition of V

(39) into two subsets. Since G is connected
Let $u \in V_1$ and $v \in V_2$. Since G is connected there exists a u-v path in G. Say,
$u = v_0, v_1, v_2 \ldots v_n = v$.

Let i be the least positive integer such that $v_i \in V_2$. Then $v_{i-1} \in V_1$ and v_{i-1}, v_i are adjacent. Thus there is an edge joining $v_{i-1} \in V_1$ and $v_i \in V_2$.

To prove the converse,

Suppose G is not connected. Then G contains at least two components. Let V_1 denote the set of all vertices of one component and V_2 the remaining vertices of G.

Clearly $V = V_1 \cup V_2$ is a partition of V and there is no edge joining any point of V_1 to any point of V_2.

Hence the theorem.

Theorem: 6 If G is not connected, then $\bar{G}$ is connected.

Proof: Since G is not connected, G has more than one component

Let u, v be any two points of $\bar{G}$. We prove that there is a u-v path in $\bar{G}$.

If u, v belong to different component in G, they are not adjacent in G and hence they are adjacent in $\bar{G}$.

If u, v lie in the same component of G, choose w in a different component. Then u, w, v is a u-v path in $\bar{G}$.

Hence $\bar{G}$ is connected.

Defn: (Distance)

For any two points u, v of a graph we define the distance between u and v by

$$d(u,v) = \begin{cases} \text{the length of a shortest } u\text{-}v \text{ path if such a path exists} \\ \infty \quad \text{otherwise} \end{cases}$$

—x—

Theorem: A graph G with at least two points is bipartite iff all its cycles are of even length.

Proof: Suppose G is a bipartite. Then V can be partitioned into two subsets V_1 and V_2 such that every edge

(41) joins a vertex of V_1 to a vertex of V_2. Now, consider any cycle $v_0, v_1, v_2 \cdots v_n = v_0$ of length n.

Suppose $v_0 \in V_1$. Then $v_2, v_4, v_6 \cdots \in V_1$ and $v_1, v_3, v_5 \cdots \in V_2$.

Further $v_n = v_0 \in V_1$ and hence n is even.

Conversely, suppose all cycles in G are of even length. We may assume without loss of generality that G is connected. Let $v_1 \in V$. Define $V_1 = \{u \in V / d(u, v_1)$ is even$\}$

$V_2 = \{u \in V / d(u, v_1)$ is odd$\}$.

Clearly, $V_1 \cap V_2 = \phi$ and $V_1 \cup V_2 = V$.

We claim that every edge of G joins a vertex of V_1 to a vertex of V_2.

Suppose two points $u, v \in V_1$ are adjacent.

Let P be a shortest $v_1 - u$ path of length m and let Q be a shortest $v_1 - v$ path of length n. Since, $u, v \in V_1$ both m and n are even.

Now, let w be the last point common to P and Q.

Then the $v_1 - w$ path along P and $v_1 - w$ path along Q are both shortest paths and hence have the same length i.

Also the $w - u$ path along P, the edge uv followed by the $v - w$ path along Q form a cycle of length $(m-i)+1+(n-i) = m+n-2i+1$ which is odd

$\Rightarrow \Leftarrow$

Thus no two points of V_1 are adjacent. Similarly no two points of V_2 are adjacent and hence G is bipartite.

Hence the theorem.

Defn: (Cutpoint)

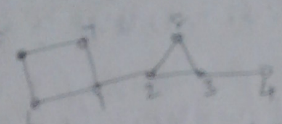
A cutpoint of a graph G is a point whose removal increases the number of components.

Defn: (Bridge (or) cut edge)

A bridge of a graph G is a edge whose removal increases the number of components

Note: If v is a cutpoint of a connected graph, $G-v$ is disconnected.

Ex:



1, 2 and 3 are cutpoints

{1,2} and {3,4} are bridges

Theorem: 1 Let v be a point of a connected graph G. The following statements are equivalent.

a) v is a cutpoint of G

b) There exists a partition of $V-\{v\}$ into subsets U and W such that for each $u \in U$ and $w \in W$, the point v is on every $u-w$ path

c) There exists two points u and w distinct from v such that v is on every $u-w$ path.

Proof: (a) ⇒ (b) Since v is a cutpoint of G, $G-v$ is disconnected. Hence $G-v$ has at least two components

Let U consists of the points of one of the components of G − v and W consists of the points of the remaining components.

Clearly, $V - \{v\} = U \cup W$ is a partition of $V - \{v\}$.

Let $u \in U$ and $w \in W$. Then u and w lie in different components of G − v. Hence there is no u−w path in G − v.

Therefore, every u−w path in G contains v.

(b) ⇒ (c). This is trivial.

(c) ⇒ (a). Since v is on every u−w path in G.

There is no u−w path in G − v. Hence G − v is not connected so that v is a cut point of G.

Theorem: 2 An edge e of a connected graph G is a bridge iff e is not on any cycle of G.

Proof: Let e be an cut edge of G → (1)

Suppose e lies on a cycle C of G

(45) (46)

Let w_1 and w_2 be any two points in G. Since, G is connected, there exists a w_1-w_2 path P in G.

If e is not on P, then P is a path in G-e.

If e is on P, replacing e by C-e, we obtain a w_1-w_2 walk in G-e.

This walk contains a w_1-w_2 path in G-e. Hence G-e is connected which is a contradiction to —(1)

Hence e is not on any cycle on G.

Conversely, Let e=uv be not on any cycle of G —(2).

Suppose e is not a bridge.

Hence, G-e is connected.

∴ There is a u-v path in G-e

This path together with the edge e=uv forms a cycle containing e which ⇒⇐ to (2).

Hence e is a bridge.

—x—

8) **Theorem:** Every non-trivial connected graph has at least two points which are not cutpoints.

Proof: Choose two points u and v such that $d(u,v)$ is maximum.

We claim that u and v are not cutpoints.

Suppose v is a cutpoint.

Hence, $G-v$ has more than one component.

~~Suppose u is a cutpoint.~~

~~Hence, $G-v$ has more than one component.~~

Choose a point w in a component that does not cutpoint u.

Then v lies on every $u-w$ path and hence, $d(u,w) > d(u,v)$ which is impossible.

Hence v is not a cutpoint.

llly, u is not a cutpoint.

Hence the theorem.

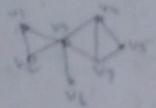
——x——

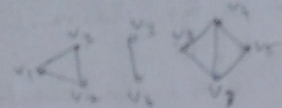
(47) (27)

Blocks

Defn: (Block)
A connected non-trivial graph having no cut point is a block.

Ex:

(G) 



(Block of G)

Theorem: Let G be a connected graph with at least three points. The following statements are equivalent.

(1) G is a block
(2) Any two points of G lie on a common cycle
(3) Any point and any line of G lie on a common cycle
(4) Any two lines of G lie on a common cycle.

Proof: (1) ⇒ (2)
Suppose G is a block.
We shall prove by induction on the distance $d(u,v)$ between u and v that any two vertices u and v lie on a common cycle.

Suppose $d(u,v)=1$. Hence u and v are adjacent. By hypothesis $G \neq K_2$ and G has no cut points. Hence the edge $e = uv$ is not a bridge and hence by theorem e is on a cycle of G.

Hence the points u and v lie on a common cycle of G.

Now assume that the result is true for any two vertices at distance less than k and let $d(u,v) = k \geq 2$.

Consider a u-v path of length k.

Let w be the vertex that precedes v on this path.

Then $d(u,w) = k-1$.

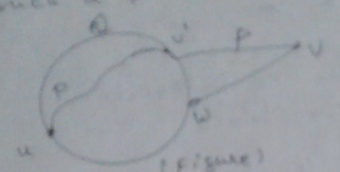
Hence by induction hypothesis there exists a cycle C that contains u and w.

Now since G is a block, w is not a cut point of G and so $G - w$ is connected.

Hence there exists a u-v path P not containing w.

Let v' be the last point common to P and C.

Since u is common to P and C. Such a v' exists.



(Figure)

Now, let Q denotes the u-v' path along the cycle C not containing the point w. Then, Q followed by the v'-v path along P, the line vw and w-u path along the cycle C lies disjoint from Q form a cycle that contains both u and v. This complete the induction.

Thus any two points of G lie on a common cycle of G.

(2) ⇒ (1). Suppose any two points of G lie on a common cycle of G.

Suppose v is a cut point of G. Then there exists two points u and w distinct from v such that every u-w path contains v.

Now, by hypothesis u and w lie on a common cycle and this cycle

(35) determined two u-w paths and at least one of these paths does not contain v ⇒

Hence G has no cutpoints.

So G is a block.

(3) ⇒ (4). Let u be a point and vw a line of G.

By hypothesis u and v lie on a common cycle C.

If w lies on C, then the line vw together with the v-w path of C contains u is the required cycle containing u and the edge vw.

If w is not on C, let C' be a cycle containing u and w.

This cycle determines two w-u paths and at least one of these paths not contain v.

Denote this path by P.

Let u' be the 1st point common to P and C. Then the edge vw followed by the w-u' subgraph of P and the u'-v path is containing u form a cycle.

Containing u and the edge uv.

(2) ⇒ (3) is trivial.

(3) ⇒ (4). The proof is same to the proof of (2) ⇒ (3)

(4) ⇒ (3) is trivial.

— x —

Connectivity

Defn: (Connectivity)

The connectivity $\kappa = \kappa(G)$ of a graph G is the minimum number of points whose removal results in a disconnected or trivial graph.

Defn: (Edge connectivity)

The edge connectivity $\lambda = \lambda(G)$ of G is the minimum number of edges whose removal results in a disconnected or trivial graph.

Ex:

(1) The connectivity and edge connectivity of a disconnected graph is 0.

(2) $\kappa(K_p) = p-1$

(52)

Theorem: For any graph G, $\kappa \le \lambda \le \delta$.

Proof: We 1st prove $\lambda \le \delta$. If G has no edges, $\lambda = \delta = 0$. Otherwise removal of all the edges incident with a point of minimum degree results in a disconnected graph. Hence $\lambda \le \delta$.

Now to prove $\kappa \le \lambda$, we consider the following cases.

Case (i): G is disconnected or trivial. Then $\kappa = \lambda = 0$.

Case (ii): G is connected graph with a bridge e. Then $\lambda = 1$. Further in this case $G = K_2$ or one of the points incident with e is a cutpoint. Hence, $\kappa = 1$ so that $\kappa = \lambda = 1$

Case (iii): $\lambda \ge 2$. Then there exist λ edges the removal of which disconnects the graph. Hence the removal of $\lambda - 1$ of these lines results in a graph G with a bridge $e = uv$. For each of these $\lambda - 1$ edges select an incident point different from u or v. The removal of these $\lambda - 1$ points removes all the $\lambda - 1$ edges.

If the resulting graph is disconnected.

then $k \leq \lambda - 1$. If not e is a bridge of this subgraph and hence the removal of u or v results in a disconnected or trivial graph.

Hence $k \leq \lambda$ and this complete the proof.

Defn: A graph G is said to be n-connected if $k(G) \geq n$.

A graph G is said to be n-edge connected if $\lambda(G) \geq n$.

Ex: (1) A non-trivial graph is 1-connected iff it is connected.

(2) K_2 is the only block which is not 2-connected.

— x —

Problem: 1 Prove that if G is a k-connected graph then $q \geq \frac{pk}{2}$.

Sln: Since G is k-connected, $k \leq \delta$

$\therefore q = \frac{1}{2} \sum d(v)$

$\geq \frac{1}{2} p\delta$ (since $d(v) \geq \delta$ for all v)

$\geq \frac{pk}{2}$.

— x —

Problem: (14) prove that there is no 3-connected graph with 7 edges.

Sol: suppose G is a k-connected graph with 7 edges.

G has 7 edges $\Rightarrow p \geq 5$.

Now, $q \geq \frac{3p}{2}$ (by problem 1)

$\therefore q \geq 15/2$.

$\therefore q \geq 8$ which is a contradiction.

Hence there is no 3-connected graph with 7 edges.

Eulerian Graphs

Defn: A closed trail containing all points and edges is called an Eulerian trail. A graph having an Eulerian trail is called an Eulerian graph.

(G)

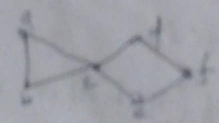


Figure: Eulerian graph

Theorem: The following statements are equivalent for a connected graph G.

(1) G is Eulerian

(2) Every point of G has even degree

(3) The set of edges of G can be partitioned into cycles.

Proof: (1) ⇒ (2): Let T be an Eulerian trail in G. with origin (and terminus) u. Each time a vertex v occurs in T in a place other than the origin and terminus, two of the edges incident with v are accounted for.

Since, an Eulerian trail contains every edges of G, d(v) is even for every v ≠ u. For u, one of the edges incident with u is accounted for by the origin of T, another by the terminus of T and others are accounted for in pairs.

Hence, d(u) is also even.

(2) ⇒ (3): Since G is connected and non-trivial every vertex of G has degree at least 2. Hence G contains a cycle Z. The removal of the edges of Z results in a spanning subgraph G_1 in which again every vertex has even degree.

If G_1 has no edges, then all the edges of G form one cycle and hence (2) holds.

Otherwise G_1 has a cycle Z_1. Removal of the edges of Z_1 from G_1 results in a spanning subgraph G_2 in which every vertex has even degree. Continuing the above process, when a graph G_n with no edge is obtained, we obtain a partition of the edges of G into n cycles.

(3) ⇒ (1): If the partition has only one cycle, then G is obviously Eulerian since it is connected. Otherwise, let $Z_1, Z_2, \ldots, Z_n$ be the cycles forming a partition of the edges of G.

Since G is connected there exists a cycle $Z_i \neq Z_1$ having a common point v_1 with Z_1. Without loss of generality, let it be Z_2. The walk beginning at v_1 and consisting of the cycles Z_1 and Z_2 in succession is a closed trail containing the edges of these two cycles. Continuing this process we can construct a closed trail containing all the edges of G. Hence G is Eulerian.

—x—

(31)

Lemma: If G is a graph in which the degree of every vertex is at least two then G contains a cycle.

proof: Construct a sequence $v_1, v_2, v_3, \ldots$ of vertices as follows. Choose any vertex v_1. Let v_2 be any vertex adjacent to v_1. Let v_3 be any vertex adjacent to v_2 other than v_1. At any stage, if vertex v_i, $i \geq 2$ is already chosen then, choose v_{i+1} to be any vertex adjacent to v_i other than v_{i-1}. Since degree of each vertex is at least 2, the existence of v_{i+1} is always guaranteed.

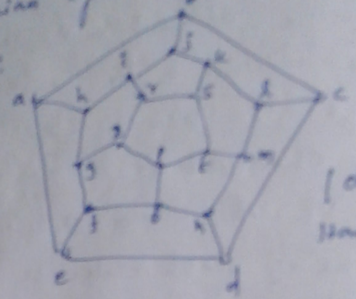
Since G has only a finite number of vertices, at some stage we have to choose a vertex which has been chosen before.

Let v_k be the first such vertex and let $v_k = v_i$ where $i < k$. Then $v_i, v_{i+1}, \ldots, v_k$ is a cycle.

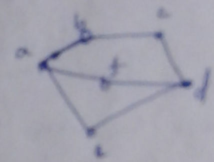
Hamiltonian

Defn: (hamiltonian graph)
A spanning cycle in a graph is called a hamiltonian cycle. A graph having a hamiltonian cycle is called a hamiltonian graph.

Ex:



(dodecahedron)
hamiltonian graph.



(Theta graph)
Non-hamiltonian graph.

— x —

Theorem: 1 Every hamiltonian graph is 2-connected.

proof: Let G be a hamiltonian graph and let Z be a hamiltonian cycle in G. For any vertex v of G, Z−v is connected and hence G−v is also connected.

Hence G has no cutpoints and thus G is 2-connected.

Theorem 2: If G is hamiltonian, then for every nonempty proper subset S of V(G), $w(G-S) \leq |S|$ where $w(H)$ denotes the no. of components in any graph H.

Proof: Let Z be a hamiltonian cycle of G. Let S be any nonempty proper subset of V(G).
Now, $w(Z-S) \leq |S|$. Also, Z-S is a spanning subgraph of G-S and hence $w(G-S) \leq w(Z-S)$. Hence $w(G-S) \leq |S|$.

— x —

Theorem 3 (Dirac)

If G is a graph with $p \geq 3$ vertices and $\delta \geq p/2$, then G is hamiltonian.

Proof: Suppose the theorem is false. Let G be a maximal non hamiltonian graph with p vertices and $\delta \geq p/2$.

Since $p \geq 3$, G cannot be complete.
Let u and v be nonadjacent vertices in G.
By the choice of G, G+uv is hamiltonian.

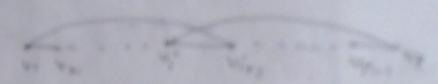
Moreover, since G is non-hamiltonian each hamiltonian cycle of $G+uv$ must contain the line uv.

Then G has a spanning path $v_1 v_2 \ldots v_p$ with origin $u=v_1$ and terminus $v=v_p$.

Let $S = \{v_i / v_1 v_{i+1} \in E\}$ and $T = \{v_i / v_i v_p \in E\}$ where E is the edge set of G.

Clearly $v_p \notin S \cup T$ and hence $|S \cup T| < p$ ——(1)

Again if $v_i \in S \cap T$, then $v_1 v_2 \ldots v_i v_p v_{p-1} \ldots v_{i+1} v_1$ is a hamiltonian cycle in G, contrary to the assumption.



(figure 1)

Hence $S \cap T = \phi$ so that $|S \cap T| = 0$ ——(2)

Also by the definition of S and T

$d(u) = |S|$ and $d(v) = |T|$.

Hence by (1) and (2), $d(u) + d(v)$

$= |S| + |T|$

$= |S \cup T| < p$

Thus $d(u)+d(v)<p$.

But since $\delta > p/2$, we have $d(u)+d(v) \geq p$

$\Rightarrow \Leftarrow$

hence the theorem.

— * —

Lemma: Let G be a graph with p points and let u and v be nonadjacent points in G such that $d(u)+d(v) \geq p$. Then G is hamiltonian iff $G+uv$ is hamiltonian.

Proof: If G is hamiltonian, then obvious-ly $G+uv$ is also hamiltonian.

Conversely, suppose that $G+uv$ is hamiltonian but G is not.

Then, as in the proof of Dirac's theorem we obtain $d(u)+d(v)<p$.

$\Rightarrow \Leftarrow$ to the hypothesis that $d(u)+d(v) \geq p$.

Thus $G+uv$ is hamiltonian

$\Rightarrow G$ is hamiltonian.

— v —

Defn: (closure)

The closure of a graph G with p points is the graph obtained from G by repeatedly joining pairs of non-adjacent vertices whose degree sum is

(63)

at least p until no such pair remains. The closure of G is denoted by $c(G)$.

Theorem: $c(G)$ is well defined.

Proof: Let G have p vertices. Let G_1 and G_2 be two graphs obtained from G by repeatedly joining pairs of nonadjacent vertices whose degree sum is at least p until no such pair remains.

Let $x_1, x_2, \ldots x_m$ and $y_1, y_2, \ldots y_n$ be the sequences of edges added to G to obtained G_1 and G_2 respectively.

We claim that $\{x_1, x_2, \ldots x_m\} = \{y_1, y_2, \ldots\}$

If possible let $x_{i+1} = uv$ be the i^{th} edge in the sequence $\{x_1, x_2, \ldots x_m\}$ that is not an edge of G_2.

Let $H = G + \{x_1, x_2, \ldots x_i\}$. Since uv is the next edge to be added to H in the process of constructing G_1, we have

$$d_H(u) + d_H(v) \geq p \quad \text{——— (1)}$$

Also by the choice of x_{i+1}, H is a subgraph of G_2.

Hence $d'(u) \geq d_H(u)$ and $d'(v) \geq d_H(v)$.

where $d'(u)$ and $d'(v)$ denote degrees of u and v in G_2.

Hence (1) together $d'(u) + d'(v) \geq p$.

Hence by the definition of G_2, u and v must be adjacent in G_2.

Since u and v are not adjacent in G_2.

Hence each x_i is an edge of G_2.

Ily we can prove that each y_i is an edge of G_1.

Hence $G_1 = G_2$. Thus $c(G)$ is unique and hence is well defined.

— x —

Theorem: A graph is hamiltonian ⟺ its closure is hamiltonian.

Proof: Let $x_1, x_2, \ldots, x_n$ be the sequence of edges added to G in obtained $c(G)$.

Let $G_1, G_2, \ldots G_n = c(G)$ be the successive graphs obtained. Applying the previous lemma

G is hamiltonian ⟺ G_1 is hamiltonian

⟺ G_2 is hamiltonian

⟺ $G_n = c(G)$ is hamiltonian

— x —

Theorem (Chvátal): Let G be a graph with degree sequence $(d_1, d_2, \ldots, d_p)$, where $d_1 \le d_2 \le \cdots \le d_p$ and $p \ge 3$. Suppose that for every value m less than $p/2$, either $d_m > m$ or $d_{p-m} \ge p-m$. Then G is hamiltonian.

Proof: Let G satisfy the hypothesis of the theorem.

We claim that $C(G)$ is complete. Let us denote the degree of a vertex v in $C(G)$ by $d'(v)$.

If possible, let $C(G)$ be not complete.

Now, let u and v be two non-adjacent vertices in $C(G)$ with

$$d'(u) \le d'(v) \quad \text{——— (1)}$$

and $d'(u) + d'(v)$ as large as possible.

Let $d'(u) = m$. Since no two non-adjacent points in $C(G)$ can have degree sum p or more, we have $d'(u) + d'(v) < p$

$$\therefore d'(v) < p - d'(u)$$

$$\therefore d'(v) < p - m \quad \text{——— (2)}$$

Now, let S denote the set of vertices in $V - \{v\}$ which are not adjacent to v in $C(G)$.

~~to ... in~~ $c(G)$.

Let T denote the set of vertices in $V-\{u\}$ which are not adjacent to u in $c(G)$.

Clearly, $|S| = p-1-d'(v)$ and
$|T| = p-1-d'(u)$ —(3)

Also by the choice of u and v, each vertex in S has degree at most $d'(u)$ and each vertex in $T \cup \{u\}$ has degree at most $d'(v)$.

putting (2) in the 1st eqn/- of (3)

we get $|S| > p-1-(p-m) = m-1$.
Hence $|S| \geq m$. Hence $c(G)$ has at least m points with degree $\leq m$ —(4).

from (3), $|T| = p-1-m$.
Since each vertex in $T \cup \{u\}$ has degree $\leq d'(u)$, this implies that $c(G)$ has at least $p-m$ vertices of degree $\leq d'(u)$.

$\therefore$ by (2) $c(G)$ has at least $p-m$ vertices of degree $< p-m$. —(5)

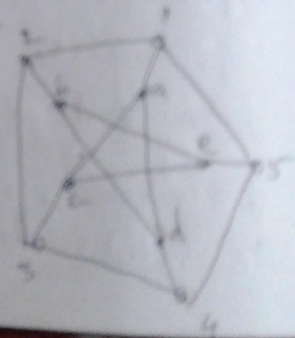
because G is a spanning subgraph of $c(G)$, degree of each point in G

cannot exceed (iii) that in $C(G)$.
Hence statements (4) & (5) hold in the case of G also.
Hence $d_m \le m$ and $d_{p-m} < p-m$.
Also by (1) and (2), $m < P/2$.
This contradicts the hypothesis on G.
$\therefore C(G)$ is complete.
Hence G is hamiltonian.

—×—

problem: Show that the petersen graph is non-hamiltonian.

Sol: Let us label the vertices as in the following figure. If the petersen graph G has a hamiltonian cycle C, then G−E(C) must be a regular spanning subgraph of degree 1 (A regular spanning subgraph of degree 1 is called a 1-factor).
Let us search for all 1-factor in G and show that none of them arise out of a hamiltonian cycle of G.



Case (1): Consider the subset $A = \{1a, 2b, 3c, 4d, 5e\}$ of the edge set of G.

Clearly A is a 1-factor of G, but G−A is the union of two disjoint cycles and hence is not a hamiltonian cycle of G.

Case (2): If the 1-factor contains 4 edges from A, then the only line passing through the remaining two points must also be included in the 1-factor, so that we again get A.

Case (3): If a 1-factor contains just 3 edges A, then two such choices can be made.

Subcase (3a): Let the 1-factor contain 1a, 2b and 3c. Now the subgraph induced by the remaining four points is a P_4 whose unique 1-factor is $\{4d, 5e\}$. Thus the 1-factor of G considered becomes A.

Subcase (3b): Let the 1-factor contain 1a, 2b and 4d. Here again the remaining four points induce P_4 whose unique 1-factor is $\{3c, 5e\}$. Thus the 1-factor consider become A.

Case (4): If a 1-factor contains just 2 edges from A.

(19)

Here again two such choices are possible.

Subcase (4a): Let the 1-factor contain 1a and 1b. In the subgraph induced by the remaining 6 points d has degree one and hence any 1-factor of that subgraph must contain edge [illegible]. Thus case 3 is repeated.

Subcase (4b): Let the 1-factor contain 1a and 1b. In the subgraph induced by the remaining 6 points [illegible] has degree one and hence any 1-factor of the subgraph must contain edge [illegible]. Thus case 3 is repeated.

Case (5): Let a 1-factor contain just one edge of A, say 1a. If it contains one more edge from A then one of the earlier cases will be repeated. Hence we have to choose the other four edges of this 1-factor from two paths each of length 3.

Hence the 1-factor is X = {1a, ce, bd, [illegible]}

Now G−X is again union of two disjoint cycles and not a hamiltonian graph.

(69)

Case(6): Suppose there exists a 1-factor that does not contain any edge from A. It can contain at most two edges from the cycle 1234 51 and at most two edges from the cycle acebda.

Hence it can contain at most four edges.

Hence there does not exist such a 1-factor.

Since, the above 6 cases cover all possible types of 1-factors, we see that G has no 1-factor arising out of a hamiltonian cycle.

Hence G has no hamiltonian cycle.

Thus G is not hamiltonian.

Matching

Defn: (Matching)

Any set M of independent edges of a graph G is called a matching of G. If $uv \in M$ we say that u and v are matched under M. We also say that the points u and v are M-saturated.

Defn: (perfect Matching)

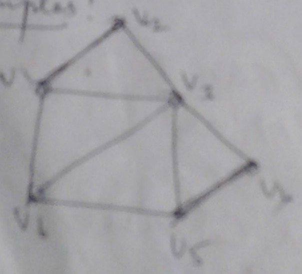
A matching M is called a perfect matching if every point of G is M-saturated.

Defn: (maximum matching)

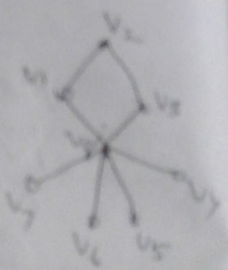
M is called a maximum matching if there is no matching M' in G such that $|M'| > |M|$.

Examples:

G_1:



G_2:



Consider the graph G_1 given in figure. $M_1 = \{v_1v_2, v_6v_3, v_5v_4\}$ is a perfect matching in G_1

$M_2 = \{v_1v_3, v_6v_5\}$ is a matching in G.

M_2 is not perfect matching.

v_2 and v_4 are not M_2 - saturated.

For the graph G_1,

$M = \{v_8v_4, v_1v_2\}$ is a maximum matching but is not a perfect matching.

— x — .

Defn: (M-alternating path)

Let M be a matching in G. A path in G is called an M-alternating path if its edges are alternately in E - M and M.

Defn: (M-augmenting path)

An M-alternating path whose origin and terminus are both M-unsaturated is called an M-augmenting path.

Ex: For the above graph G_1,

$P_1 = (v_6, v_5, v_4, v_8)$ is an M-alternating path.

$P_2 = (v_2, v_1, v_3, v_6, v_5, v_4)$ is an M_2-augmenting path.

p is even. However the converse is not true. The graph G_2 given in Fig 7.1 has an even number of vertices but has no perfect matching.

Theorem 7.1. *Let M_1 and M_2 be two matchings in a graph G. Let $M_1 \Delta M_2 = (M_1 - M_2) \cup (M_2 - M_1)$ be the symmetric difference of M_1 and M_2. Let $H = G[M_1 \Delta M_2]$ be the subgraph of G induced by $M_1 \Delta M_2$. Then each component of H is either an even cycle with edges alternately in M_1 and M_2 or a path P with edges alternately in M_1 and M_2 such that the origin and the terminus of P are unsaturated in M_1 or M_2.*

Proof. Let v be any point in H. Since M_1 and M_2 are matchings in G, at most one line of M_1 and at most one line of M_2 are incident with v. Hence the degree of v in H is either 1 or 2. Hence it follows that the components of H must be as described in theorem.

Example. For the graph G_2 given in Fig 7.1

$$M_1 \Delta M_2 = \{v_1v_2, [illegible]\}$$

the graph $H_2 = G_2[M_1 \Delta M_2]$ is given in Fig 7.2.

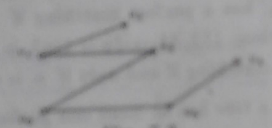


Fig. 7.2

Clearly H_2 is a path whose edges are alternately in M_1 or M_2. The origin v_2 and terminus v_4 are both M_2-unsaturated.

The following theorem due to Berge gives a characterisation of maximum matching.

Theorem 7.2. *A matching M in a graph G is a maximum matching if and only if G contains no M-augmenting path.*

Proof. Let M be a maximum matching in G. Suppose G contains an M-augmenting path $P = (v_0, v_1, v_2, \cdots, v_{2k+1})$.

By definition of M-augmenting path the lines $v_0v_1, v_2v_3, \cdots, v_{2k}v_{2k+1}$ are not in M and the lines $v_1v_2, v_3v_4, \cdots, v_{2k-1}v_{2k}$ are in M. Hence

$$M' = M - \{v_1v_2, v_3v_4, \cdots, v_{2k-1}v_{2k}\} \cup \{v_0v_1, v_2v_3, \cdots, v_{2k}v_{2k+1}\}$$

is a matching in G and $|M'| = |M| + 1$, which is a contradiction, since M is a maximum matching. Hence G has no M-augmenting path. Conversely,

(58)

suppose G has no M-augmenting path. If M is not a maximum matching in G then there exists a matching M' of G such that $|M'| > |M|$. Let $H = G[M \Delta M']$. By theorem 5.1, each component of H is either an even cycle with edges alternately in M and M' or a path P with edges alternately in M and M' such that the origin and the terminus of P are unsaturated in M or M'. Clearly any component of H which is a cycle contains equal number of edges from M and M'. Since $|M'| > |M|$ there exists at least one component of H which is a path whose first and last edges are from M'. Thus the origin and terminus of P are M'-saturated in H and hence they are M-unsaturated in G. Thus P is an M-augmenting path in G, which is a contradiction. Hence M is a maximum matching in G.

Solved Problems

Problem 1. For what values of n does the complete graph K_n have perfect matching.

Solution. Clearly any graph with p odd has no perfect matching. Also the complete graph K_n has a perfect matching if n is even. For example if $V(K_n) = \{1, 2, \cdots, n\}$ then $\{12, 34, \cdots, (n-1)n\}$ is a perfect matching of K_n. Thus K_n has a perfect matching if and only if n is even.

Problem 2. Show that a tree has at most one perfect matching.

Solution. Let T be a tree. Suppose T has two perfect matchings say M_1 and M_2. Then degree of every vertex in $H = T[M_1 \Delta M_2]$ is 2. Hence every component of H is an even cycle with edges alternately in M_1 and M_2. This is a contradiction, since T has no cycles. Therefore T has at most one perfect matching.

Problem 3 Find the number of perfect matchings in the complete bipartite graph $K_{n,n}$.

Solution. Let $A = \{x_1, x_2, \cdots, x_n\}$ and $B = \{y_1, y_2, \cdots, y_n\}$ be a bipartition of $K_{n,n}$.

We observe that any matching of $K_{n,n}$ that saturates every vertex of A is a perfect matching. Now the vertex x_1 can be saturated in n ways by choosing any one of the edges $x_1y_1, x_1y_2, \cdots, x_1y_n$. Having saturated x_1, the vertex x_2 can be saturated in $n-1$ ways. In general having saturated $x_1, x_2, \cdots, x_i$ the next vertex x_{i+1} can be saturated in $n-i$ ways. Hence the number of perfect

Consider the graph G_1 given in figure.

6 TREES

6.0 INTRODUCTION

In this chapter we study a special class of graphs, known as trees, which arise from the study of operators in differential calculus. The concept of a tree was discovered by Cayley in 1857, who related his work on trees to the study of chemical composition. Trees are very important for the sake of their applications to many different fields. Further a tree is the simplest non-trivial type of a graph and in trying to prove a general result or test a conjecture in Graph Theory, it is sometimes convenient to first study the situation for trees.

6.1 CHARACTERISATION OF TREES

Definition. A graph that contains no cycles is called an **acyclic graph**. A connected acyclic graph is called a **tree**. Any graph without cycles is also called a **forest** so that the components of a forest are trees.

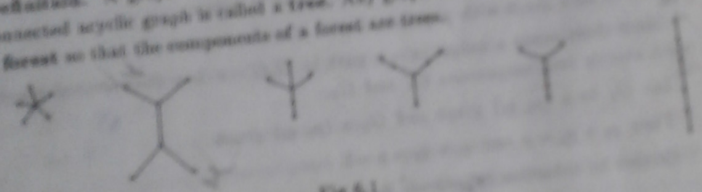


Fig 6.1

Fig 6.1 gives all trees with 6 vertices.

The following theorem gives several equivalent ways of describing a tree.

Theorem 6.1. *Let G be a (p,q) graph. The following statements are equivalent.*

1. *G is a tree.*
2. *Every two points of G are joined by a unique path.*
3. *G is connected and $p = q + 1$.*
4. *G is acyclic and $p = q + 1$.*

Proof. $1 \Rightarrow 2$: Let u, v be any two points of G.

Since G is connected there exists a $u - v$ path in G.

Now suppose there exist two distinct $u - v$ paths.

$$P_1 : u = v_0, v_1, v_2, \ldots v_n = v \text{ and}$$

$P_1 = (v_6, v_5, v_4, v_3)$ is an M – alternating path.

$$P_2 : u = w_0, w_1, w_2, \ldots, w_m = v.$$

Let i be the least positive integer such that $1 \leq i \leq m$ and $w_i \notin P_1$. (Such an i exists since P_1 and P_2 are distinct).

Hence $w_{i-1} \in P_1 \cap P_2$.

Let j be the least positive integer such that $i < j \leq m$ and $w_j \in P_1$. Then the $w_{i-1} - w_j$ path along P_2 followed by the $w_j w_{i-1}$ path along P_1 form a cycle which is a contradiction.

Hence there exists a unique $u - v$ path in G.

$2 \Rightarrow 3$: Clearly G is connected.

We prove $p = q + 1$ by induction on p.

This is trivial for a connected graph with 1 or 2 points.

Assume the result for graphs with fewer than p points.

Let G be a graph with p points. Let $x = uv$ be any line of G.

Since there exists a unique $u - v$ path in G, $G - x$ is a disconnected graph with exactly two components G_1 and G_2.

Let G_1 be a (p_1, q_1) graph and G_2 a (p_2, q_2) graph.

Then $p_1 + p_2 = p$ and $q_1 + q_2 = q - 1$.

Further by induction hypothesis $p_1 = q_1 + 1$ and $p_2 = q_2 + 1$.

$$\begin{aligned} \text{Hence } p &= p_1 + p_2 \\ &= q_1 + q_2 + 2 \\ &= q - 1 + 2 \\ &= q + 1. \end{aligned}$$

$3 \Rightarrow 4$: We must prove that G is acyclic.

Suppose G contains a cycle of length n.

There are n points and n lines on this cycle. Fix a point u on the cycle. Consider any one of the remaining $p - n$ points not on the cycle, say v.

Since G is connected we can find a shortest $u - v$ path in G. Consider the line on this shortest path incident with v. The $p - n$ lines thus obtained are all distinct.

Consider the graph G_1 given in figure.

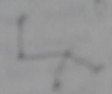
Hence $q \geq (p-n)+n = p$ which is a contradiction since $q+1=p$. Thus G is acyclic.

(⇐ 1) Since G is acyclic, to prove that G is a tree we need only to prove that G is connected.

Suppose G is not connected. Let $G_1, G_2, \ldots, G_k (k \geq 2)$ be the components of G.

Since G is acyclic each of these components is a tree.

Hence $q_i + 1 = p_i$ where G_i is a (p_i, q_i) graph.



$\therefore \sum_{i=1}^{k}(q_i+1) = \sum_{i=1}^{k} p_i$

i.e., $q + k = p$ and $k \geq 2$, which is a contradiction.

Hence G is connected.

This completes the proof.

Corollary. *Every non-trivial tree G has at least two vertices of degree 1.*

Proof. Since G is non-trivial, $d(v) \geq 1$ for all points v. Also $\sum d(v) = 2q = 2(p-1) = 2p-2$.

Hence $d(v) = 1$ for at least two vertices.

Theorem 6.2. *Every connected graph has a spanning tree.*

Proof. Let G be a connected graph. Let T be a minimal connected spanning subgraph of G. Then for any line x of T, $T - x$ is disconnected and hence x is a bridge of T.

Hence T is acyclic.

Further T is connected and hence is a tree.

Corollary. *Let G be a (p,q) connected graph. Then $q \geq p-1$.*

Proof. Let T be a spanning tree of G. Then the number of lines in T is $p-1$. Hence $q \geq p-1$.

Theorem 6.3. *Let T be a spanning tree of a connected graph G. Let $x = uv$ be an edge of G not in T. Then $T + x$ contains a unique cycle.*

Proof. Since T is acyclic every cycle in $T + x$ must contain x. Hence there exists a one to one correspondence between cycles in $T + x$ and $u - v$ paths in T. As there is a unique $u - v$ path in tree T, there is a unique cycle in $T + x$.

$P_1 = (v_6, v_5, v_4, v_3)$ is an M-alternating path.

$P_2 = (v_2, v_1, v_3, v_6, v_5, v_4)$ is an M_2 alternating path.

8 PLANARITY

8.0 INTRODUCTION

In this chapter we consider the embedding (drawing without crossings) of graphs on surfaces, especially the plane. We study the properties of planar graphs and state Kuratowski's famous theorem, on the characterization of planar graphs. The materials in this chapter form part of what is known as topological graph theory.

8.1 DEFINITION AND PROPERTIES

Definition. A graph is said to be **embedded** in a surface S when it is drawn on S so that no two edges intersect ("meeting" of edges at a vertex is not considered an intersection). A graph is called **planar** if it can be drawn on a plane without intersecting edges. A graph is called **non-planar** if it is not planar. A graph that is drawn on the plane without intersecting edges is called a **plane graph**.

Example. The graph in Fig.8.1(a) is planar even though it is not plane.

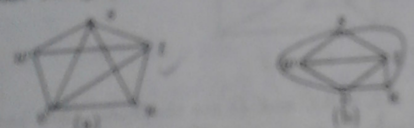


Fig. 8.1

The graph in Fig. 8.1(b) (which is isomorphic to that in Fig.8.1 (a)) is plane as it is drawn without intersecting edges. It is also planar. Thus plane graph is a concept associated with embedding of the graph.

It is obvious that if two graphs are isomorphic and one is planar, then the other is also planar. However, as is seen from Fig.8.1, if two graphs are isomorphic and one is plane, the other need not be plane.

Theorem 8.1. K_5 is non-planar.

Proof. If possible, let K_5 be planar. K_5 contains a cycle of length five say (s, t, u, v, w, s).

Hence, without loss of generality, any plane embedding of K_5 can be assumed to contain this cycle drawn in the form of a regular pentagon [See Fig. 8.1(b)]

Hence the edge uv must lie either wholly inside the pentagon or wholly outside it.

Suppose that uv is wholly inside the pentagon. (The argument when it lies wholly outside the pentagon is quite similar). Since the edge xz and xu do not cross the edge uv, they must both lie outside the pentagon. The edge vt cannot cross the edge xu. Hence vt must be inside the pentagon. But now, the edge wy crosses one of the edges already drawn, giving a contradiction. Hence K_5 is non-planar.

Definition. Let G be a graph embedded on a plane π. Then $\pi - G$ is the union of disjoint regions. Such regions are called **faces** of G. Each plane graph has exactly one unbounded face and it is called the **exterior face**. Let F be a face of a plane graph G and e be an edge of G. Let P be a point in F. e is said to be in the **boundary** of F if for every point Q of e on e there exists a curve joining P and Q which lies entirely in F.

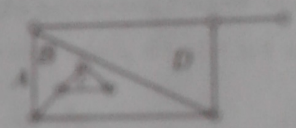


Fig. 8.2

For the plane graph in Fig. 8.2, A,B,C and D are the faces. They have 5, 7, 3 and 3 edges respectively in their boundary. A is the exterior face of G.

Theorem 8.2. *A graph can be embedded in the surface of a sphere iff it can be embedded in a plane.*

Proof. Let G be a graph embedded on a sphere. Place the sphere on a plane L and call the point of contact S (south pole). At point S, draw a normal to the plane and let N (north pole) be the point where this normal intersects the surface of the sphere.

Assume that the sphere is placed in such a way that N is disjoint from G.

For each point P on the sphere, let P' be the unique point on the plane where the line NP intersects the surface of the plane. Thus there is a one-to-one correspondence between the points of the sphere other than N and the points on the plane.

Hence (1) gives $p - q + r + k - 1 = 2k$ so that $p - q + r = k + 1$ as required.

Corollary 2. *If G is a (p,q) plane graph in which every face is an n-cycle then* $q = \frac{n(p-2)}{n-2}$.

Proof. Every face is an n-cycle. Hence each edge lies on the boundary of exactly two faces. Let $f_1, f_2, \cdots, f_r$ be the faces of G.

$2q = \sum_{i=1}^{r}$ (number of edges in the boundary of face f_i) $= nr$.

$r = 2q/n$.

By Euler's formula, $p - q + r = 2$.

$$p - q + \frac{2q}{n} = 2.$$

$$q\left(\frac{2}{n} - 1\right) = 2 - p.$$

$$q = \frac{n(p-2)}{n-2}.$$

Corollary 3. *In any connected plane (p,q) graph $(p \geq 3)$ with r faces $q \geq 3r/2$ and $q \leq 3p - 6$.*

Proof. Case 1. Let G be a tree.

Then $r = 1, q = p - 1$ and $p \geq 3$. Hence $q \geq 3r/2$ and $q \leq 3p - 6$ since $p - 1 \leq 3p - 6$ (as $p \geq 3$).

Case 2. Let G have a cycle.

Let f_i, ($i = 1$ to r) be the faces of G. Since each edge lies on the boundary of atmost two faces,

$2q \geq \sum_{i=1}^{r}$ (number of edges in the boundary of face f_i)

i.e., $2q \geq 3r$ since each face is bounded by at least three edges. $\cdots$ (1)

i.e. $q \geq 3r/2$

By Euler's formula, $p - q + r = 2$.

Substituting for r in (1), we get $q \geq \frac{3}{2}(2 + q - p)$ which on simplification gives $q \leq 3p - 6$.

Definition. A graph is called **maximal planar** if no line can be added to it [illegible] without losing planarity. In a maximal planar graph, each face is a triangle and such a graph is sometimes called a **triangulated graph.**

The following corollary follows directly from corollary 3 and the fact that in a maximal planar graph every face is a triangle.

Corollary 4. *If G is a maximal planar (p,q) graph then $q = 3p - 6$.*

Corollary 5. *If G is a plane connected (p,q) graph without triangles and $p \geq 3$ then $q \leq 2p - 4$.*

Proof. If G is a tree, then $q = p - 1$. Hence we have $p - 1 = q \leq 2p - 4$ (since $3 \leq p$). Now let G have a cycle. Since G has no triangles, the boundary of each face has at least four edges. Since each edge lies on at most two faces we have $2q \geq \sum_{i=1}^{r}$ (number of edges in the boundary of the i^{th} face).

i.e., $2q \geq 4r$ ... (4)

But $p - q + r = 2$ by Euler's formula.

Substituting for r in (4), we get $2q \geq 4(2 + q - p)$.

Hence $4p - 8 \geq 2q$ so that $q \leq 2p - 4$.

Corollary 6. *The graphs K_5 and $K_{3,3}$ are not planar.*

Proof. K_5 is a $(5,10)$ graph.

For any planar graph, $q \leq 3p - 6$ by Corollary 3.

But $q = 10$ and $p = 5$ do not satisfy this inequality.

Hence K_5 is not planar.

$K_{3,3}$ is a $(6,9)$ bipartite graph and hence has no triangles. If such a graph is planar, then by Corollary 5, $q \leq 2p - 4$.

But $p = 6$ and $q = 9$ do not satisfy this inequality. Hence $K_{3,3}$ is not planar.

Corollary 7. *Every planar graph G with $p \geq 3$ points has at least three points of degree less than 6.*

Proof. By corollary 3, $q \leq 3p - 6$.

i.e., $2q \leq 6p - 12$.

i.e., $\sum d_i \leq 6p - 12$ where d_i are the degrees of the vertices of G. Since G is connected, $d_i \geq 1$ for every i. If at most two d_i are less than 6, then $\sum d_i \geq 1 + 1 + 6 + \cdots (p-2)$ times $= 6p - 10$ which contradicts (1).

Hence $d_i < 6$ for at least three values of i.

Theorem 8.7. *Every planar graph G with at least 3 points is a subgraph of a triangulated graph with the same number of points.*

Proof. Let G have p vertices. If $p \leq 4$, then G must be a subgraph of K_4 which is a triangulated (maximal planar) graph. Hence let $p \geq 5$.

We construct a triangulated graph G' which contains G as a subgraph as follows:

Consider a plane embedding of G. If R is a face of G and v_1 and v_2 are two vertices on the boundary of R without a connecting edge we connect v_1 and v_2 with an edge lying entirely in R. This yields a new plane graph. This operation is continued until every pair of vertices on the boundary of the same face are connected by an edge. The number of vertices remains the same under these operations and hence the process terminations after some time yielding a plane triangulated graph G'. G is obviously a subgraph of G'.

This theorem is of great use in the following sense. To prove "4-colourability" for every planar graph, one common approach often used is to prove 4-colourability for maximal planar graphs (triangulations) as it is rather easier to deal with maximal planar graphs than with arbitrary planar graphs.

Exercises.

1. Show that $K_5 - e$ is planar for every edge e.
2. Show that $K_{3,3} - e$ is planar for every edge e.
3. Redraw the graph in Fig.8.2 with B as the infinite face.
4. If G is a connected (p, q) plane graph with girth g, then $q \leq \frac{g(p-2)}{g-2}$.
5. Prove that there is no 6-connected planar graph.
6. State true or false: A graph is planar iff every proper subgraph is planar.

8.2 CHARACTERIZATION OF PLANAR GRAPHS

To decide whether a given graph G is planar or not is an important problem. We discuss some criterion for planarity below. It is clear that a disconnected graph is planar iff each of its components is planar. Also a graph is planar iff all its blocks are planar. Therefore, while considering embedding or planarity, it is enough if only 2-connected graphs are considered. It is obvious that a graph

$P_1 = (v_6, v_5, v_4, v_3)$ is an M-alternating

9 COLOURABILITY

9.0 INTRODUCTION

For more than 120 years, four colour conjecture (4CC) was the most famous unsolved problem in mathematics. Attempts to settle the 4CC led to a lot of results and a number of generalisations of the 4CC. In this chapter, we discuss the theory of colourings, which grew along with attempts to settle the 4CC. This includes the proof of the five colour theorem, which may be regarded as the most beautiful result proved in this book.

9.1 CHROMATIC NUMBER AND CHROMATIC INDEX

Definition. An assignment of colours to the vertices of a graph so that no two adjacent vertices get the same colour is called a **colouring** of the graph. For each colour, the set of all points which get that colour is independent and is called a **colour class**. A colouring of a graph G using at most n colours is called an n **colouring**. The **chromatic number** $\chi(G)$ of a graph G is the minimum number of colours needed to colour G. A graph G is called, n-**colourable** if $\chi(G) \leq n$.

Example.

Graph	K_p	$K_p - e$	$\overline{K}_p$	$K_{m,n}$	C_{2n}	C_{2n+1}
Chromatic Number	p	$p-1$	1	2	2	3

When T is a tree with atleast two points, $\chi(T) = 2$.

A wheel has chromatic number 3 or 4 according as it has an odd or even number of points.

Definition. Each n-colouring of G partitions $V(G)$ into n independent sets called colour classes. Such a partitioning induced by a $\chi(G)$ colouring of G is called a chromatic partitioning. In other words, a partition of $V(G)$ into the smallest possible number of independent sets is called a chromatic partitioning of G.

Example. $\{1, 4, 8\}, \{3, 6, 7\}, \{2, 5\}$ is a chromatic partitioning of the graph in Fig.9.1. which has chromatic number 3.

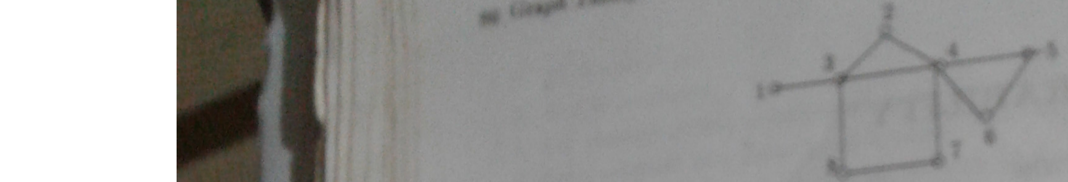


Fig. 9.1.

Theorem 9.1. *The following statements are equivalent for any graph G.*

i. *G is 2-colourable.*
ii. *G is bipartite.*
iii. *Every cycle of G has even length.*

Proof. (i) $\Rightarrow$ (ii). G is 2-colourable. Hence $V(G)$ can be partitioned into two colour classes. These colour classes are independent sets and hence form a bipartition of G. Hence G is a bipartite.

(ii) $\Rightarrow$ (i). G is bipartite. Hence $V(G)$ can be partitioned into two sets V_1 and V_2 such that V_1 and V_2 are independent sets. A 2-colouring of G can be obtained by colouring all the points of V_1 white and all the points of V_2 blue. Hence G is 2-colourable. (ii) $\Leftrightarrow$ (iii) follows from theorem 4.7.

Remark. G is bipartite does not imply $\chi(G) = 2$. For example $\overline{K}_2$, which is bipartite, has chromatic number 1. However, if G has an edge and is bipartite, then $\chi(G) = 2$.

Definition. A graph G is called **critical** if $\chi(H) < \chi(G)$ for every proper subgraph H of G. A $k-$ chromatic graph that is critical is called $k-$ **critical**. It is obvious that every $k-$ chromatic graph has a $k-$ critical subgraph.

Theorem 9.2. *If G is $k-$ critical, then $\delta(G) \geq k-1$.*

Proof. Since G is $k-$ critical, for any vertex v of G, $\chi(G-v) = k-1$. If deg $v < k-1$, then a $(k-1)-$ colouring of $G-v$ can be extended to a $k-1$ colouring of G by assigning to v, a colour which is assigned to none of its neighbours in G. Hence deg $v \geq k-1$, so that $\delta(G) \geq k-1$.

Corollary 1. *Every k-chromatic graph has at least k vertices of degree at least $k-1$.*

Proof. Let G be a k-chromatic graph and H be a k- critical subgraph of G. By Theorem 9.2, $\delta(H) \geq k-1$. Also since $\chi(H) = k$, H has at least k vertices.

Hence H has at least k vertices of degree at least $k-1$. Since H is a subgraph of G, the result follows.

Corollary 2. *For any graph G, $\chi \leq \Delta + 1$.*

Proof. Let G have chromatic number χ. Let H be a χ-critical subgraph of G. By Theorem 9.2, $\delta(H) \geq \chi - 1$. Hence $\chi \leq \delta(H) + 1$. Since $\delta(H) \leq \Delta(G)$, this implies that $\chi \leq \Delta(G) + 1$.

Theorem 9.3. *For any graph G, $\chi(G) \leq 1 + \max \delta(G')$ where the maximum is taken over all induced subgraphs G' of G.*

Proof. The theorem is obvious for totally disconnected graphs. Now let G be an arbitrary n-chromatic graph, $n \geq 2$. Let H be any smallest induced subgraph of G such that $\chi(H) = n$. Hence $\chi(H - v) = n - 1$ for every point v of H. If $\deg_H v < n - 1$, then a $(n-1)$-colouring of $H - v$ can be extended to a $n-1$ colouring of H by assigning to v, a colour which is assigned to none of its neighbours in H. Hence $\deg_H v \geq n - 1$. Since v is an arbitrary vertex of H, this implies that $\delta(H) \geq n - 1 = \chi(G) - 1$.

Hence $\chi(G) \leq 1 + \delta(H) \leq 1 + \max \delta(H')$ where the maximum is taken over the set A of induced subgraphs H' of H. Hence $\chi(G) \leq 1 + \max \delta(G')$, where the maximum is taken over the set B of induced subgraphs G' of G.

Definition. If $\chi(G) = n$ and every n-colouring of G induces the same partition on $V(G)$ then G is called **uniquely** n-colourable or **uniquely colourable**. $K_4 - x$, K_3 and $K_4 - x$ are uniquely 3-colourable. K_n is uniquely n-colourable. $K_n - x$ is uniquely $(n-1)$ colourable. Any connected bipartite graph is uniquely 2-colourable.

Theorem 9.4. *If G is uniquely n-colourable, then $\delta(G) \geq n - 1$.*

Proof. Let v be any point of G. In any n-colouring, v must be adjacent with at least one point of every colour different from that assigned to v. (Otherwise, by recolouring v with a colour which none of its neighbours is having, a different n-colouring can be achieved. Hence degree of v is at least $n-1$ so that $\delta(G) \geq n - 1$.

Theorem 9.5. *Let G be a uniquely n-colourable graph. Then in any n-colouring of G, the subgraph induced by the union of any two colour classes is connected.*

Proof. If possible, let C_1 and C_2 be two classes in a n-colouring of G such that the subgraph induced by $C_1 \cup C_2$ is disconnected. Let H be a component of the subgraph induced by $C_1 \cup C_2$. Obviously, no point of H is adjacent to a point in $V(G) - V(H)$ that is coloured C_1 or C_2. Hence interchanging the colours of the points in H and retaining the original colours for all other points we get a different n-colouring for G. This gives a contradiction.

Note. This type of interchange of colours in a subgraph is used often in the study of colourings.

Theorem 9.6. *Every uniquely n-colourable graph is $(n-1)$-connected.*

Proof. Let G be a uniquely n-colourable graph. Consider an n-colouring of G. If possible, let G be not $(n-1)$-connected. Hence there exists a set S of at most $n-2$ points such that $G-S$ is either trivial or disconnected. If $G-S$ is trivial, then G has at most $n-1$ points so that G is not uniquely n-colourable. Hence $G-S$ has at least two components. In the considered n-colouring, there are at least two colours say c_1 and c_2 that are not assigned to any point of S.

If every point in a component of $G-S$ has colour different from c_1 and c_2, then by assigning colour c_1 to a point of this component, we get a different n-colouring of G. Otherwise, by interchanging the colours c_1 and c_2 in a component of $G-S$, a different n-colouring of G is obtained. In any case, G is not uniquely n-colourable, giving a contradiction.

Hence G is $(n-1)$ connected.

Corollary. *In any n-colouring of a uniquely n-colourable graph G, the subgraph induced by the union of any k colour classes, $2 \le k \le n$, is $(k-1)$-connected.*

Proof. If the subgraph H induced by the union of any k colour classes, $2 \le k \le n$, had different k-colourings, then these k-colourings will induce different n-colourings for G giving a contradiction. Hence H is uniquely k-colourable. Hence by Theorem 9.6 H is $(k-1)$-connected.

Definition. An assignment of colours to the edges of a graph G so that no two adjacent edges get the same colour is called an **edge colouring** or **line colouring** of G. An edge colouring of G using n colours is called a n-edge colouring (or n-line colouring). The **edge chromatic number** (also called **line chromatic number** or **chromatic index**) $\chi'(G)$ is the minimum number of

Hence $\chi'(K_n) \leq n-1$. Also $\chi'(K_n) \geq \Delta(K_n) = n-1$.

Hence $\chi'(K_n) = n-1$.

Exercises.

1. Give an example of a graph with $\Delta = \chi'$ and a graph with $\Delta < \chi'$.
2. Show that every outerplanar graph is 3-colourable.
3. What is the smallest uniquely 3-colourable graph?
4. What is the smallest uniquely 3-colourable graph which is not complete?
5. Show that for any independent set S of points of a critical graph G, $\chi(G - S) = \chi(G) - 1$.
6. Show that the Petersen graph has chromatic index 4.

9.2 THE FIVE COLOUR THEOREM

Heawood (1890) showed that one can always colour the vertices of a plane graph with at most five colours. This is known as the five-colour theorem.

Theorem 9.9. *Every planar graph is 5-colourable.*

Proof. We will prove the theorem by induction on the number p of points. For any planar graph having $p \leq 5$ points, the result is obvious since the graph is p-colourable.

Now let us assume that all planar graphs with p points is 5-colourable for some $p \geq 5$. Let G be a planar graph with $p+1$ points. Then G has a vertex v of degree 5 or less. (Corollary 7 to theorem 7.6). By induction hypothesis, the plane graph $G - v$ is 5-colourable. Consider a 5-colouring of $G - v$ where $c_i, 1 \leq i \leq 5$, are the colours used. If some colour, say c_j is not used in colouring vertices adjacent to v, then by assigning the colour c_j to v the 5-colouring of $G - v$ can be extended to a 5-colouring of G.

Hence we have to consider only the case in which $\deg v = 5$ and all the five colours are used for colouring the vertices of G adjacent to v.

Let v_1, v_2, v_3, v_4, v_5 be the vertices adjacent to v coloured c_1, c_2, c_3, c_4 and c_5 respectively.

Let G_{13} denote the subgraph of $G - v$ induced by those vertices coloured c_1 or c_3. If v_1 and v_3 belong to different components of G_{13}, then a 5-colouring of $G - v$ can be obtained by interchanging the colours of vertices in the component of G_{13} containing v_1. (Since no point of this component is adjacent to a point with colour c_1 or c_3 outside this component, this interchange of colours results is a colouring of $(G - v)$. In this 5-colouring no vertex adjacent to v is coloured c_1, and hence by colouring v with c_1, a 5-colouring of G is obtained.

If v_1 and v_3 are in the same component of G_{13}, then in G there exists a $v_1 - v_3$ path all of whose points are coloured c_1 or c_3. Hence there is no $v_2 - v_4$ path all whose points are coloured c_2, c_4 (Fig. 9.2).

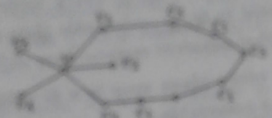


Fig. 9.2

Hence if G_{24} denotes the subgraph of $G - v$ induced by the points coloured c_2 or c_4, then v_2 and v_4 belong to different components of G_{24}. Hence if we interchange the colours of the points in the component of G_{24} containing v_2, a new 5-colouring $G - v$ results and in this, no point adjacent to v is coloured c_2. Hence by assigning colour c_2 to v, we can get a 5-colouring of G. This completes the induction and the proof.

9.3 FOUR COLOUR PROBLEM

The four colour conjecture states that any map on a plane or on the surface of a sphere can be coloured with only four colours so that no two adjacent countries have the same colour. Each country must consist of a single connected region and adjacent countries are those having a boundary line (not merely a single point) in common. The problem of deciding whether the four colour conjecture is true or false is called the four colour problem.

As seen in section 7.2 a plane graph (geometric dual) can be associated with each map. Colouring the countries of the map is equivalent to colouring the vertices of its geometric dual. In this set up, the four colour conjecture states that "Every planar graph is 4-colourable".

The number 4 cannot be reduced further as there are maps that require at least four colours. The map in fig 9.3 is one such map.

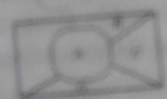


Fig. 9.3

There are several problems in graph theory that are equivalent to the four colour problem. One of these is the case $n = 5$ of the following conjecture.

Unit - V

So to verify the proof, one has to believe the computer. For this reason, many people hesitate to use the following.

Theorem 9.11 (Four colour theorem). *Every planar graph is 4-colourable.*

A computer-free proof of the above theorem is still to be found.

Definition. Let S be an orientable surface. The chromatic number of S, denoted by $\chi(S)$ is defined to be $\chi(S) = \max\{\chi(G) / G$ is a graph embeddable on $S\}$.

Let S_n denote the orientable surface of genus n. (S_n is topologically equivalent to a sphere with n handles and S_0 is the ordinary sphere).

Theorem 9.12 (Heawood map colouring Theorem). *For every positive integer* n, $\chi(S_n) = \left[\frac{7 + \sqrt{1 + 48n}}{2}\right]$.

The final proof of this theorem was given by Ringel and Youngs. Note that four colour theorem is the extension of the above theorem to the case $n = 0$.

9.4 CHROMATIC POLYNOMIALS

Birkhoff (1912) introduced chromatic polynomials as a possible means of attacking the four colour conjecture. This concept considers the number of ways of colouring a graph with given number of colours.

Let G be a labeled graph. A colouring of G from λ colours is a colouring of G which uses λ or fewer colours. Two colourings of G from λ colours will be considered different if at least one of the labeled points is assigned different colours. Let $f(G, \lambda)$ denote the number of different colourings of G from λ colours.

For Example, $f(K_1, \lambda) = \lambda$ and $f(\overline{K}_2, \lambda) = \lambda^2$.

Theorem 9.13. $f(K_n, \lambda) = \lambda(\lambda - 1) \cdots (\lambda - n + 1)$.

Proof. The first vertex in K_n can be coloured in λ different ways (as there are λ colours). For each colouring of the first vertex, the second vertex can be coloured in $\lambda - 1$ ways (as there are $\lambda - 1$ colours remaining). For each colouring of the first two vertices, the third can be coloured in $\lambda - 2$ ways and so on.

Hence $f(K_n, \lambda) = \lambda(\lambda - 1) \cdots (\lambda - n + 1)$.

Remark. $f(\overline{K}_n, \lambda) = \lambda^n$, since each of the n points of $\overline{K}_n$ may be coloured independently in λ ways.

$P_1 = (v_6, v_5, v_4, v_3)$ is an M-alternating path.

Theorem 9.14. *If G is a graph with k components $G_1, G_2, \ldots, G_k$ then*

$$f(G,\lambda) = \pi_{i=1}^{k} f(G_i,\lambda).$$

Proof. Number of ways of colouring G_i with λ colours is $f(G_i,\lambda)$. Since any choice of λ-colourings for $G_1, \ldots, G_k$ can be combined to give a λ-colouring for G, $f(G,\lambda) = \pi_{i=1}^{k} f(G_i,\lambda)$.

Definition. Let u and v be two nonadjacent points in a graph G. The graph obtained from G by the removal of u and v and the addition of a new point w adjacent to those points to which u or v was adjacent is called an **elementary homomorphism** of G. In other words, identification of two nonadjacent points of G is called an elementary homomorphism.

Theorem 9.15. *If u and v are nonadjacent points in a graph G and hG denotes the elementary homomorphism of G which identifies u and v, then $f(G,\lambda) = f(G + uv, \lambda) + f(hG,\lambda)$ where $G + uv$ denotes the graph obtained from G by adding the line uv.*

Proof. $f(G,\lambda)$ = number of colouring of G from λ colours.

= (number of colourings G from λ colours in which u and v get different colours) + (number of colourings of G from λ colours in which u and v get the same colour.)

= (number of colourings of $G + uv$ from λ colours) + (number of colourings of hG from λ colours)

$= f(G + uv, \lambda) + f(hG, \lambda)$.

Corollary.

(i) *For any graph G, $f(G,\lambda)$ is a polynomial in λ.*

(ii) *$f(G,\lambda)$ has degree $|V(G)|$.*

(iii) *The constant term in $f(G,\lambda)$ is 0.*

Proof. The above theorem, states that $f(G,\lambda)$ can be written as the sum of $f(G_1,\lambda)$ and $f(G_2,\lambda)$ where G_1 has the same number of points as G with one more edge and G_2 has one point less than G. Doing this process repeatedly $f(G,\lambda)$ can be written as $\sum f(G_i,\lambda)$ where each G_i is a complete graph and $\max|V(G_i)| = |V(G)|$.

Since $f(K_n,\lambda)$ is a polynomial of degree n, it follows that $f(G,\lambda)$ is a polynomial of degree $|V(G)|$. Since $f(K_n,\lambda)$ has constant term 0, the constant term in $\sum f(G_i,\lambda)$ is 0 so that (iii) holds.

Note. Because of the above corollary $f(G, \lambda)$ is called the **chromatic polynomial** of G.

The chromatic polynomial of a graph can be determined using Theorem 9.13 as illustrated in the following solved problem.

Solved problem

Problem 1. Find the chromatic polynomial of the graph G given in Fig. 9.4.

Solution. A diagram of the graph is used to denote its chromatic polynomial. The nonadjacent points considered at each step are indicated by u and v.

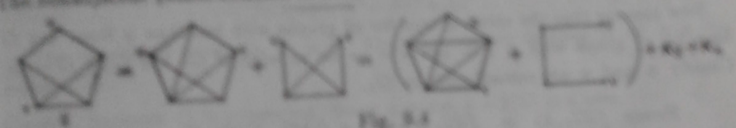


Fig. 9.4

$$\begin{aligned} \text{Thus } f(G, \lambda) &= [(K_5 + K_4) + (K_4 + K_3)] + (K_4 + K_3) \\ &= K_5 + 3K_4 + 2K_3 \\ &= f(K_5, \lambda) + 3f(K_4, \lambda) + 2f(K_3, \lambda) \\ &= \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4) + \\ &\quad 3\lambda(\lambda-1)(\lambda-2)(\lambda-3) + 2\lambda(\lambda-1)(\lambda-2) \\ &= \lambda^5 - 7\lambda^4 + 19\lambda^3 - 23\lambda^2 + 10\lambda. \end{aligned}$$

Theorem 9.16. *If G is a tree with n points, $n \geq 2$, then $f(G, \lambda) = \lambda(\lambda-1)^{n-1}$.*

Proof. We prove the result by induction on n. For $n = 2$, $G = K_2$ and hence $f(G, \lambda) = f(K_2, \lambda) = \lambda(\lambda-1)$ so that the theorem holds. Assume that the chromatic polynomial of any tree with $n-1$ points is $\lambda(\lambda-1)^{n-2}$. Let G be a tree with n points. Let v be an end point of G and let u be the unique point of G adjacent to v. By hypothesis, the tree $G - v$ has $\lambda(\lambda-1)^{n-2}$ for its chromatic polynomial. The point u can be assigned any colour different from that assigned to u. Hence v may be coloured in $\lambda - 1$ ways for each colouring of $G - v$. Thus $f(G, \lambda) = (\lambda-1)f(G - v, \lambda) = (\lambda-1)\lambda(\lambda-1)^{n-2} = \lambda(\lambda-1)^{n-1}$.

This complete the induction and the proof.

The converse of the above theorem is also true as given below.

Theorem 9.17. *A graph G with n points and $f(G, \lambda) = \lambda(\lambda-1)^{n-1}$ is a tree.*

$P_1 = (v_6, v_5, v_4, v_3)$ is an M-alternating path.

All the trees in Fig. 8.1 have the chromatic polynomial $\lambda(\lambda-1)^4$. But they are not isomorphic. In the following solved problems we give some more properties of chromatic polynomials.

Solved problems.

Problem 1. Prove that the coefficients of $f(G,\lambda)$ alternate in sign.

Solution. We prove the result by induction on the number of lines q. When $q=0$, $f(G,\lambda)=\lambda^p$ where p is the number of points of G. In this case the polynomial has just one non-zero coefficient and hence the result is trivially true.

Now assume that the result is true for all graphs with less than q lines. Let G be a (p,q) graph with $q>0$. Let $x=uv$ be an edge of G. Let $G_1=G-uv$. Clearly u and v are nonadjacent in G_1.

$$\text{Hence } f(G_1,\lambda) = f(G_1+uv,\lambda)+f(hG_1,\lambda) \text{ (by Theorem 8.13)}$$
$$= f(G,\lambda)+f(hG_1,\lambda).$$

Hence $f(G,\lambda)=f(G_1,\lambda)-f(hG_1,\lambda)$ ... (1)

Now G_1 is a $(p,q-1)$ graph and hG_1 is a $(p-1,q_1)$ graph where $q_1<q$. Hence by induction hypothesis,

$$f(G_1,\lambda) = \lambda^p-\alpha_1\lambda^{p-1}+\alpha_2\lambda^{p-2}-\cdots+(-1)^{p-1}\alpha_{p-1}\lambda$$
$$\text{and } f(hG_1,\lambda) = \lambda^{p-1}-\beta_1\lambda^{p-2}+\cdots+(-1)^{p-2}\beta_{p-2}\lambda$$

where α_i and β_j are non-negative integers.

Hence by (1)

$$f(G,\lambda)=\lambda^p-(\alpha_1+1)\lambda^{p-1}+(\alpha_2+\beta_1)\lambda^{p-2}-\cdots+(-1)^{p-1}(\alpha_{p-1}+\beta_{p-2})\lambda.$$

This is a polynomial in which the coefficients alternate in sign.

This completes the induction and the proof.

Problem 2. Prove that if G is a (p,q) graph, the coefficient of λ^{p-1} in $f(G,\lambda)$ is $-q$.

Solution. We prove the result by induction on q. If $q=0$ the $f(G,\lambda)=\lambda^p$. Hence the coefficient of $\lambda^{p-1}=0=-q$.

Now assume the result for all graphs with less than q edges.

As in the previous problem,

$$f(G,\lambda)=f(G_1,\lambda)-f(hG_1,\lambda) \qquad (1)$$

Since G_1 is a $(p, q-1)$ graph by induction hypothesis coefficient of λ^{p-1} in $f(G_1, \lambda) = -(q-1)$.

Also, coefficient of λ^{p-1} in $f(hG_1, \lambda) = 1$.

Hence, coefficient of λ^{p-1} in $f(G, \lambda) = -(q-1) - 1 = -q$ (by using 1).

This complete the induction and the proof.

Problem 3. Prove that $\lambda^4 - 3\lambda^3 + 3\lambda^2$ cannot be the chromatic polynomial of any graph.

Solution. Suppose there exists a graph G such that

$$f(G, \lambda) = \lambda^4 - 3\lambda^3 + 3\lambda^2.$$

The number of points in G is 4. Also the number of lines in G is 3 (by problem 2).

Case 1. Suppose G is connected. Since $q = 3 = p - 1$, G is a tree.

$$\text{Hence} \quad f(G, \lambda) = \lambda(\lambda - 1)^3 \text{ (By Theorem 9.16)}$$
$$= \lambda^4 - 3\lambda^3 + 3\lambda^2 - \lambda$$

which is a contradiction.

Case 2. Suppose G is not connected.

Then $G = K_3 \cup K_1$. $\quad \therefore f(G, \lambda) = f(K_3, \lambda) f(K_1, \lambda)$

$$= [\lambda(\lambda - 1)(\lambda - 2)]\lambda$$
$$= \lambda^4 - 3\lambda^3 + 2\lambda^2$$

which is again a contradiction.

Hence the result is proved.

Exercises.

1. Find the chromatic polynomials of

(a) $K_4 - x$ where x is a line.

(b) $\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$

$P_1 = (v_6, v_5, v_4, v_3)$ is an M-alternating path.

$P_2 = (v_2, v_1, v_3, v_6, v_5, v_4)$ is an M-

10 DIRECTED GRAPHS

10.0 INTRODUCTION

In this chapter, we give the basic theory of digraphs. First we give some basic properties of digraphs followed by a discussion of paths and the three types of connectedness. We discuss the conditions under which one can direct the edges of a graph in such a way that the resulting digraph is strongly connected. Then we deal with the connection between digraphs and matrices. Finally we give some properties of tournaments and introduce Stockmeyer's tournaments.

10.1 DEFINITIONS AND BASIC PROPERTIES

Definition. A **directed graph** (or in short **digraph**) D is a pair (V, A) where V is a finite nonempty set and A is a subset of $V \times V - \{(v,v)/v \in V\}$. The elements of V and A are respectively called **vertices** (**points**) and **arcs**. If $(u,v) \in A$ then the arc (u,v) is said to have u as its initial vertex (tail) and v as its terminal vertex (head). Also the arc (u,v) is said to join u to v.

Remark. Just as graphs, digraphs can also be represented by means of diagrams. In those diagrams, vertices are denoted by points and the arc (u,w) is represented by means of an arrow from u to w. We shall often refer to the diagram of a digraph as the digraph itself.

Example. $D = (\{1,2,3,4\}, \{(1,2),(2,3),(1,3),(3,1)\})$ is a digraph. The diagram representing D is given in Fig. 10.1.

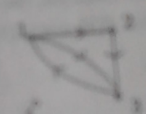


Fig. 10.1

Definition. The **indegree** $d^-(v)$ of a vertex v in a digraph D is the number of arcs having v as its terminal vertex. The **outdegree** $d^+(v)$ of v is the number of arcs having v as its initial vertex. The ordered pair $(d^+(v), d^-(v))$ is called the **degree pair** of v.

The degree pairs of the points 1, 2, 3, and 4 of the digraph in Fig 10.1 are $(2,1), (1,1), (1,2)$, and $(0,0)$ respectively.

Theorem 10.1. *In a digraph D, sum of the indegrees of all the vertices is equal to the sum of their out degrees, each sum being equal to the number of arcs in D.*

Proof. Let q denote the number of arcs in $D = (V, A)$.

$P_1 = (v_6, v_5, v_4, v_3)$ is an M-alternating path.
$P_2 = (v_2, v_1, v_3, v_6, v_5, v_4)$ is an M-

Let $B = \sum_{v \in V} d^+(v)$ and $C = \sum_{v \in V} d^-(v)$.

An arc (v, w) contributes one to the outdegree of v and one to the indegree of w. Hence each arc contributes 1 to the sum B and 1 to the sum C. Hence $B = C = q$.

Definition. A digraph $D' = (V', A')$ is called a **subdigraph** of $D = (V, A)$ if $V' \subseteq V$ and $A' \subseteq A$. The definition of **induced subdigraph** is analogous to that of induced subgraph. The **underlying graph** G of a digraph D is a graph having the same vertex set as D and two vertices v and w are adjacent in G whenever (v, w) or (w, v) is in A.

For example the digraph $D = (\{1, 2, 3, 4\}, \{(1, 2), (3, 4), (4, 2), (3, 2), (1, 4), (4, 1), (2, 4)\})$, has $K_4 - e$ as its underlying graph.

Similarly if we are given a graph G we can obtain a digraph from G by giving orientation to each edge of G. A digraph thus obtained from G is called an orientation of G.

Definition. Two digraphs $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ are said to be **isomorphic** (written $D_1 \cong D_2$) if there exists a bijection $f : V_1 \to V_2$ such that $(v, w) \in A_1$ iff $(f(v), f(w)) \in A_2$. f is called an **isomorphism** from D_1 to D_2.

For example, the digraphs in Fig. 10.2 are isomorphic under the mapping f where $f(1) = a, f(2) = b, f(3) = c, f(4) = d$.

Fig. 10.2

Theorem 10.2. *If two digraphs are isomorphic then corresponding points have the same degree pair.*

Proof. Let $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ be isomorphic under an isomorphism f. Let $v \in V_1$.

Let $N(v) = \{w / w \in V_1 \text{ and } (v, w) \in A_1\}$ and

$N(f(v)) = \{w / w \in V_2 \text{ and } (f(v), f(w)) \in A_2\}$

Now, $w \in N(v) \Leftrightarrow (v, w) \in A_1$

$\Leftrightarrow (f(v), f(w)) \in A_2$ (since f is an isomorphism)

$\Leftrightarrow f(w) \in N(f(v))$ (by definition of $N(f(v))$)

Hence $|N(v)| = |N(f(v))|$ (since f is a bijection.)

Here the LHS and RHS are respectively the outdegree of v and $f(v)$. Hence v and $f(v)$ have the same outdegree.

Similarly we can prove that v and $f(v)$ have the same indegree and hence v and $f(v)$ have the same degree pair.

Because of Theorem 10.1 and 10.2, it is obvious that two isomorphic digraphs have the same number vertices and the same number of arcs.

Definition. The **converse** digraph D' of a digraph D is obtained from D by reversing the direction of each arc.

Obviously D and D' have same number of points and arcs. Moreover, the indegree of a point v in D is equal to its outdegree in D' and vice versa.

Definition. A digraph $D = (V, A)$ is called **complete** if for every pair of distinct points v and w in V, both (v, w) and (w, v) are in A.

Thus if a complete digraph has n vertices then it has $n(n-1)$ arcs.

Definition. A digraph is called **functional** if every point has outdegree 1.

If a functional digraph has n vertices then the sum of the outdegrees of the points is n. Hence by Theorem 10.1 the number of arcs in the digraph is n. (We will study more on functional digraphs in Chapter 10).

Exercises.

1. Show that the digraph in Figures 10.1 and 10.2 are not isomorphic.
2. Draw the diagram for the digraph $D = \{a, b, c, d, e\}$, $\{(a, c), (a, d), (b, e)(e, c), (d, e)\}$. Write down the degree pair of each point of D. Find the converse D' of D and find the degree pair of each point D'.

10.2 PATHS AND CONNECTIONS

Definition. A **walk (directed walk)** in a digraph is a finite alternating sequence $W = v_0 x_1 v_1 \cdots x_n v_n$ of vertices and arcs in which $x_i = (v_{i-1}, v_i)$ for every arc x_i. W is called a walk from v_0 to v_n or a $v_0 - v_n$ walk. The vertices v_0 and v_n are called **origin** and **terminus** of W respectively and $v_1, v_2, \cdots v_{n-1}$ are

$P_1 = (v_6, v_5, v_4, v_3)$ is an M-alternating path.

called its **internal vertices**. The **length** of a walk is the number of occurrence of arcs in it. A walk in which the origin and terminus coincide is called a **closed walk**.

A **path** (**directed path**) is a walk in which all the vertices are distinct. A **cycle** (**directed cycle** or **circuit**) is a nontrivial closed walk whose origin and internal vertices are distinct.

If there is a path from u to v then v is said to be **reachable** from u. A digraph is called **strongly connected** or **diconnected** or **strong** if every pair of points are mutually reachable. A digraph is called **unilaterally connected** or **unilateral** if for every pair of points, at least one is reachable from the other. A digraph is called **weakly connected** or **weak** if the underlying graph is connected. A digraph is called **disconnected** if the underlying graph is disconnected.

The trivial digraph consisting just one point is (vacuously) strong since it does not contain two distinct points. Obviously, strongly connected ⇒ unilaterally connected ⇒ weakly connected. But the converse is not true.

Theorem 10.3. *The edges of a connected graph $G = (V, E)$ can be oriented so that the resulting digraph is strongly connected iff every edge of G is contained in at least one cycle.*

Proof. Suppose the edges of G can be oriented so that the resulting digraph becomes strongly connected.

If possible, let $e = uw$ be an edge of G not lying on any cycle. Now, as soon as e is oriented, one of the vertices u and w becomes non-reachable from the other. Hence an orientation of the required type is not possible, giving contradiction. Hence every edge of G lies on a cycle.

Conversely, let every edge of G lie on a cycle.

Let $S = v_1 v_2 \cdots v_k v_1$ be a cycle in G. Orient the edges of S so that S becomes a directed cycle and hence becomes a strongly connected subdigraph. If $V = \{v_1, \cdots, v_k\}$ then we are through. Otherwise, let w be a vertex of G not in S such that w is adjacent to a vertex v_i of S. Let $e = v_i w$. By hypothesis, e lies on some cycle C. We choose a direction of C and give the orientation determined by this direction to the edges of C which are not already oriented. The resulting enlarged oriented graph is also strongly connected as it can be got from S by a sequence of additions of simple directed paths. (For example, if $v \in S$ and w is a point on a simple directed $v_1 - v_2$ path P added to S then in the enlarged oriented graph the $w - v_2$ subpath of P followed by the $v_2 - v$ subpath of S give a directed $w - v$ path. Also, the $v - v_1$ subpath of S followed by the $v_1 - w$ subpath of P give a directed $v - w$ path. This type of argument can be repeated for each addition of simple, directed paths.) (See Fig. 10.3.)

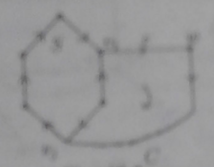


Fig. 10.3

This process can be repeated till we get a strongly connected oriented spanning subgraph of G. The remaining edges can now be oriented in any way. The resulting oriented graph is strongly connected. This completes the proof.

There are three different kinds of components of a digraph.

Definition. Let $D = (V, A)$ be a digraph.

(a) Let W_1 be a maximal subset of V such that for every pair of points $u, v \in W_1$, u is reachable from v and v is reachable from u. Then the subdigraph of D induced by W_1 is called a **strong component** of D.

(b) Let W_2 be a maximal subset of V such that for every pair points $u, v \in W_2$, either u is reachable from v or v is reachable from u. Then the subdigraph of D induced by W_2 is called a **unilateral component** of D.

(c) Let W_3 be a maximal subset of V such that for every pair of points $u, v \in W_3$, u and v are joined by a path in the underlying graph of D. Then the subdigraph of D induced by W_3 is called a **weak component** of D.

Note. Let D be a digraph. Then each point of D is in exactly one strong component of D. An arc x lies in exactly one strong component if it lies on a cycle. There is no strong component containing an arc that does not lie on any cycle.

Example. Consider the digraph D is given Fig. 10.4. The strong components are those subdigraphs induced by the sets of points $A = \{1, 2\}$;

$P_1 = (v_6, v_5, v_4, v_3)$ is an M-alternating path.

$P_2 = (v_2, v_1, v_3, v_6, v_5, v_4)$ is an M-